MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 11 January 2021 2–6 PM

TEACHER: Patrik Albin 031 7723512 palbin@chalmers.se .

AIDS: All aids are permitted. (See the Canvas course "Ordinarie tentamen Modul: 0107, MVE170" with instructions for this reexam for clarifications.)

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(k)\}_{k\in\mathbb{Z}}$ be a discrete time WSS random process. Is it true in general that $Y(k)=\sin((X(k))^2)$, $k\in\mathbb{Z}$, is also a WSS process? **(5 points)**

Task 2. Consider a time discrete $\mathbb{Q}$-valued ($\mathbb{Q}$ being the rational numbers) Markov chain $\{X_k\}_{k=0}^{\infty}$. Let $Y_n=\sum_{k=0}^{n} h(n-k)\,X_k$ for $n\geq 0$ for some function $h:\mathbb{N}\to\mathbb{Q}\setminus\{0\}$. Is it true in general that $\{Y_n\}_{n=0}^{\infty}$ is also a Markov chain? **(5 points)**

Task 3. Let (X,Y) be a continuous random variable with probability density function $f_{X,Y}(x,y)$. Prove in two different ways that $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\}=\mathbf{E}\{\mathbf{1}_A(X)Y\}$ for the indicator $\mathbf{1}_A(X)=1$ when $X\in A$ and $\mathbf{1}_A(X)=0$ when $X\notin A$ of a set $A\subseteq\mathbb{R}$.

(5 points)

Task 4. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda>0$ and B a Binomial$(1,1/2)$ random variable that is independent of $\{N(t)\}_{t\geq 0}$. For which functions $f(t)$ (if any) is $M(t)=f(t)\,(-7)^{B+N(t)}$, $t\geq 0$, a martingale with respect to the filtration $F_s=\sigma(M(r):r\in[0,s])$, $s\geq 0$? **(5 points)**

Task 5. A continuous time Markov chain $\{X(t)\}_{t\geq 0}$ has state space $\{-1,1\}$, generator $G=\left(\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}\right)$ and starting distribution $\mu^{(0)}=\pi$ the stationary distribution. Calculate the characteristic function for the random variable $(X(s),X(s+t))$ for $s,t\geq 0$.

(5 points)

Task 6. Let $\{X(t)\}_{t\geq 0}$ denote the total number of customers in a M(1)/M(1)/1 queueing system (with $\lambda=\mu=s=1$ and $K=\infty$). Find the probability distribution of the time $T=\min\{t\geq 0: X(t)=2\}$ when $X(0)=0$. **(5 points)**

MSG800/MVE170 Solutions to exam 11 January 2021

Task 1. Let $X(k)=N_k$ for k even and $X(k)=e_k$ for k odd where $\{N_k\}$ and $\{e_k\}$ are independent random variables with N_k $\mathrm{N}(0,1)$-distributed and e_k a random sign (1 or -1 with probability 1/2 each). Then $\{X(k)\}$ is discrete time white noise and thus WSS. However $\mathbf{E}\{\sin(X(k)^2)\}=\int_{-\infty}^{\infty}\sin(x^2)\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx\approx 0.351578$ for k even while $\mathbf{E}\{\sin(X(k)^2)\}=\sin(1)\approx 0.841471$ for k odd so that $\{\sin(X(k)^2)\}$ is not WSS.

Task 2. As $h\neq 0$, to know the history of the Y-process $Y_n,\ldots,Y_0$ is same thing as know the history of the X-process $X_n,\ldots,X_0$. But $\mathbf{P}\{Y_{n+1}=j|Y_n,\ldots,Y_0\}=\mathbf{P}\{h(0)X_{n+1}+\sum_{k=0}^{n}h(n+1-k)\,X_k=j|X_n,\ldots,X_0\}$ which is not the same as $\mathbf{P}\{h(0)X_{n+1}+\sum_{k=0}^{n}h(n+1-k)\,X_k=j|Y_n\}$ as all $X_n,\ldots,X_0$ affect the event $\{h(0)X_{n+1}+\sum_{k=0}^{n}h(n+1-k)\times X_k=j\}$. So the answer in no!

Task 3. First proof using properties 4 and 7 on page 220 in Hsu: $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\}=\mathbf{E}\{\mathbf{E}\{\mathbf{1}_A(X)Y|X\}\}=\mathbf{E}\{\mathbf{1}_A(X)Y\}$.

Second elementry proof: $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\}=\int_{-\infty}^{\infty}\mathbf{1}_A(x)\mathbf{E}\{Y|X=x\}\,f_X(x)\,dx=\int_{-\infty}^{\infty}\mathbf{1}_A(x)\big(\int_{-\infty}^{\infty}yf_{Y|X}(y|x)\,dy\big)f_X(x)\,dx=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{1}_A(x)y\,f_{X,Y}(x,y)\,dxdy=\mathbf{E}\{\mathbf{1}_A(X)Y\}$.

Task 4. $\mathbf{E}\{M(t)|F_s)=f(t)\,(-7)^{B+N(s)}\,\mathbf{E}\{(-7)^{N(t)-N(s)}\}=f(t)\,(-7)^{B+N(s)}\,\mathrm{e}^{-8\lambda(t-s)}$ for $0\leq s\leq t$ giving $f(t)=C\,\mathrm{e}^{8\lambda t}$ for some constant $C\in\mathbb{R}$.

Task 5. As $\mu^{(s)}=\pi=(\frac{1}{2}\ \frac{1}{2})$, $(P_t)_{i,i}=\frac{1}{2}(1+\mathrm{e}^{-2t})$ and $(P_t)_{i,j}=\frac{1}{2}(1-\mathrm{e}^{-2t})$ for $i\neq j$ [cf. G&S Exercise 6.9.1], we have

$$
\begin{aligned}
&\Psi_{X(s),X(s+t)}(\omega_1,\omega_2)\\
&=\mathbf{E}\big\{\mathrm{e}^{j(\omega_1X(s)+\omega_2X(s+t))}\big\}\\
&=\pi_1\big((P_t)_{11}\mathrm{e}^{j(\omega_1+\omega_2)}+(P_t)_{1,-1}\mathrm{e}^{j(\omega_1-\omega_2)}\big)+\pi_{-1}\big((P_t)_{-1,1}\mathrm{e}^{j(\omega_2-\omega_1)}+(P_t)_{-1,-1}\mathrm{e}^{-j(\omega_1+\omega_2)}\big)\\
&=\ldots=\cos(\omega_1)\cos(\omega_2)-\mathrm{e}^{-2t}\sin(\omega_1)\sin(\omega_2).
\end{aligned}
$$

Task 6. Writing $T_{1\to 2}$ for the time it takes to move from state 1 to state 2 and using that $\Psi_{\exp(\lambda)}(\omega)=\frac{\lambda}{\lambda-j\omega}$ we have

$$\Psi_T(\omega)=\frac{1}{1-j\omega}\,\Psi_{1\to 2}(\omega)\quad\text{and}\quad\Psi_{1\to 2}(\omega)=\frac{2}{2-j\omega}\Big(\frac{1}{2}\,\Psi_T(\omega)+\frac{1}{2}\,\Psi_0(\omega)\Big)$$

giving

$$\Psi_T(\omega)=\frac{1}{1-3j\omega-\omega^2}=\frac{1}{\sqrt{5}}\,\frac{1}{(\frac{3}{2}-\frac{\sqrt{5}}{2})-j\omega}-\frac{1}{\sqrt{5}}\,\frac{1}{(\frac{3}{2}+\frac{\sqrt{5}}{2})-j\omega}$$

which by taking inverse CHF yields PDF $f_T(t)=\frac{1}{\sqrt{5}}\,\mathrm{e}^{-(\frac{3}{2}-\frac{\sqrt{5}}{2})t}-\frac{1}{\sqrt{5}}\,\mathrm{e}^{-(\frac{3}{2}+\frac{\sqrt{5}}{2})t}$ for $t\geq 0$.