MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 11 January 2021 2–6 PM

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AIDS: All aids are permitted. (See the Canvas course "Ordinarie tentamen Modul: 0107, MVE170" with instructions for this reexam for clarifications.)

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(k)\}_{k\in\mathbb{Z}}$ be a discrete time WSS random process. Is it true in general that $Y(k) = \sin((X(k))^2)$, $k \in \mathbb{Z}$, is also a WSS process? **(5 points)**

Task 2. Consider a time discrete $\mathbb{Q}$-valued ($\mathbb{Q}$ being the rational numbers) Markov chain $\{X_k\}_{k=0}^{\infty}$. Let $Y_n = \sum_{k=0}^{n} h(n-k)\, X_k$ for $n \geq 0$ for some function $h : \mathbb{N} \to \mathbb{Q}\setminus\{0\}$. Is it true in general that $\{Y_n\}_{n=0}^{\infty}$ is also a Markov chain? **(5 points)**

Task 3. Let (X, Y) be a continuous random variable with probability density function $f_{X,Y}(x, y)$. Prove in two different ways that $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\} = \mathbf{E}\{\mathbf{1}_A(X)Y\}$ for the indicator $\mathbf{1}_A(X) = 1$ when $X \in A$ and $\mathbf{1}_A(X) = 0$ when $X \notin A$ of a set $A \subseteq \mathbb{R}$.

(5 points)

Task 4. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and B a Binomial$(1, 1/2)$ random variable that is independent of $\{N(t)\}_{t\geq 0}$. For which functions $f(t)$ (if any) is $M(t) = f(t)\,(-7)^{B+N(t)}$, $t \geq 0$, a martingale with respect to the filtration $F_s = \sigma(M(r) : r \in [0, s])$, $s \geq 0$? **(5 points)**

Task 5. A continuous time Markov chain $\{X(t)\}_{t\geq 0}$ has state space $\{-1, 1\}$, generator $G = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ and starting distribution $\mu^{(0)} = \pi$ the stationary distribution. Calculate the characteristic function for the random variable $(X(s), X(s+t))$ for $s, t \geq 0$.

(5 points)

Task 6. Let $\{X(t)\}_{t\geq 0}$ denote the total number of customers in a M(1)/M(1)/1 queueing system (with $\lambda = \mu = s = 1$ and $K = \infty$). Find the probability distribution of the time $T = \min\{t \geq 0 : X(t) = 2\}$ when $X(0) = 0$. **(5 points)**

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Task 1. Let $X(k) = N_k$ for k even and $X(k) = e_k$ for k odd where $\{N_k\}$ and $\{e_k\}$ are independent random variables with N_k $\mathrm{N}(0,1)$-distributed and e_k a random sign (1 or -1 with probability 1/2 each). Then $\{X(k)\}$ is discrete time white noise and thus WSS. However $\mathbf{E}\{\sin(X(k)^2)\} = \int_{-\infty}^{\infty} \sin(x^2) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, dx \approx 0.351578$ for k even while $\mathbf{E}\{\sin(X(k)^2)\} = \sin(1) \approx 0.841471$ for k odd so that $\{\sin(X(k)^2)\}$ is not WSS.

Task 2. As $h \neq 0$, to know the history of the Y-process $Y_n, \ldots, Y_0$ is same thing as know the history of the X-process $X_n, \ldots, X_0$. But $\mathbf{P}\{Y_{n+1} = j | Y_n, \ldots, Y_0\} = \mathbf{P}\{h(0)X_{n+1} + \sum_{k=0}^{n} h(n+1-k)\, X_k = j | X_n, \ldots, X_0\}$ which is not the same as $\mathbf{P}\{h(0)X_{n+1} + \sum_{k=0}^{n} h(n+1-k)\, X_k = j | Y_n\}$ as all $X_n, \ldots, X_0$ affect the event $\{h(0)X_{n+1} + \sum_{k=0}^{n} h(n+1-k) \times X_k = j\}$. So the answer in no!

Task 3. First proof using properties 4 and 7 on page 220 in Hsu: $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\} = \mathbf{E}\{\mathbf{E}\{\mathbf{1}_A(X)Y|X\}\} = \mathbf{E}\{\mathbf{1}_A(X)Y\}$.

Second elementry proof: $\mathbf{E}\{\mathbf{1}_A(X)\mathbf{E}\{Y|X\}\} = \int_{-\infty}^{\infty} \mathbf{1}_A(x)\mathbf{E}\{Y|X=x\}\, f_X(x)\, dx = \int_{-\infty}^{\infty} \mathbf{1}_A(x)\big(\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy\big) f_X(x)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{1}_A(x) y\, f_{X,Y}(x,y)\, dxdy = \mathbf{E}\{\mathbf{1}_A(X)Y\}$.

Task 4. $\mathbf{E}\{M(t)|F_s\} = f(t)\,(-7)^{B+N(s)}\,\mathbf{E}\{(-7)^{N(t)-N(s)}\} = f(t)\,(-7)^{B+N(s)}\,\mathrm{e}^{-8\lambda(t-s)}$ for $0 \le s \le t$ giving $f(t) = C\,\mathrm{e}^{8\lambda t}$ for some constant $C \in \mathbb{R}$.

Task 5. As $\mu^{(s)} = \pi = (\frac{1}{2}\ \ \frac{1}{2})$, $(P_t)_{i,i} = \frac{1}{2}(1+\mathrm{e}^{-2t})$ and $(P_t)_{i,j} = \frac{1}{2}(1-\mathrm{e}^{-2t})$ for $i \neq j$ [cf. G&S Exercise 6.9.1], we have

$$
\begin{aligned}
&\Psi_{X(s),X(s+t)}(\omega_1,\omega_2)\\
&= \mathbf{E}\big\{\mathrm{e}^{j(\omega_1 X(s)+\omega_2 X(s+t))}\big\}\\
&= \pi_1\big((P_t)_{11}\mathrm{e}^{j(\omega_1+\omega_2)} + (P_t)_{1,-1}\mathrm{e}^{j(\omega_1-\omega_2)}\big) + \pi_{-1}\big((P_t)_{-1,1}\mathrm{e}^{j(\omega_2-\omega_1)} + (P_t)_{-1,-1}\mathrm{e}^{-j(\omega_1+\omega_2)}\big)\\
&= \ldots = \cos(\omega_1)\cos(\omega_2) - \mathrm{e}^{-2t}\sin(\omega_1)\sin(\omega_2).
\end{aligned}
$$

Task 6. Writing $T_{1\to 2}$ for the time it takes to move from state 1 to state 2 and using that $\Psi_{\exp(\lambda)}(\omega) = \frac{\lambda}{\lambda - j\omega}$ we have

$$
\Psi_T(\omega) = \frac{1}{1-j\omega}\,\Psi_{1\to 2}(\omega) \quad \text{and} \quad \Psi_{1\to 2}(\omega) = \frac{2}{2-j\omega}\Big(\frac{1}{2}\,\Psi_T(\omega) + \frac{1}{2}\,\Psi_0(\omega)\Big)
$$

giving

$$
\Psi_T(\omega) = \frac{1}{1-3j\omega-\omega^2} = \frac{1}{\sqrt{5}}\,\frac{1}{(\frac{3}{2}-\frac{\sqrt{5}}{2})-j\omega} - \frac{1}{\sqrt{5}}\,\frac{1}{(\frac{3}{2}+\frac{\sqrt{5}}{2})-j\omega}
$$

which by taking inverse CHF yields PDF $f_T(t) = \frac{1}{\sqrt{5}}\,\mathrm{e}^{-(\frac{3}{2}-\frac{\sqrt{5}}{2})t} - \frac{1}{\sqrt{5}}\,\mathrm{e}^{-(\frac{3}{2}+\frac{\sqrt{5}}{2})t}$ for $t \ge 0$.