MSG800/MVE170 Basic Stochastic Processes

Written home re-exam Wednesday 7 April 2021 8.30–12.30

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AIDS: All aids are permitted. (See the Canvas course "Omtentamen 1 Modul: 0107, MVE170" with instructions for this reexam for clarifications.)

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(t)\}_{t\geq 0}$ be a Poisson process with rate 1, $\{Y(t)\}_{t\geq 0}$ a zero-mean WSS Gaussian process with autocorrelation function $R_Y(\tau) = \mathrm{e}^{-|\tau|}$ that is independent of the X-process and S and T independent unit mean exponential distributed random times that are independent of the X- and Y-processes. Find an expression for $\mathbf{P}\{X(S)Y(T) > x\}$ for $x > 0$. [HINT: Recall that $\int_0^\infty z^k \,\mathrm{e}^{-z}\, dz = k!$ for $k \in \mathbb{N}$.] **(5 points)**

Task 2. Let $N(t)$ be the number of customers at time $t \geq 0$ in an M/M/1/3 queueing system with $\lambda = \mu = 1$ such that $N(0) = 0$. Find by means of stochastic simulation an approximation of the expected value of the time $T = \min\{t > 0 : N(t) = 0,\ N(s) = 3 \text{ for some } s \in (0, t)\}$ it takes to move from empty queueing system to full queueing system and back to empty queueing system again. (Analytic solutions give zero points.)

(5 points)

Task 3. Kal and Ada start with initial fortune 2 trillion SEK each and repeatedly play a game where Karl wins with probability $p \in (0, 1)$ and Ada wins with probability $1 - p$ and where the winner gets 1 trillion SEK from the looser after each game. The gaming goes on until one of Kal and Ada has no money left and the other one of them is declared the winner. Find the probability that Kal wins. **(5 points)**

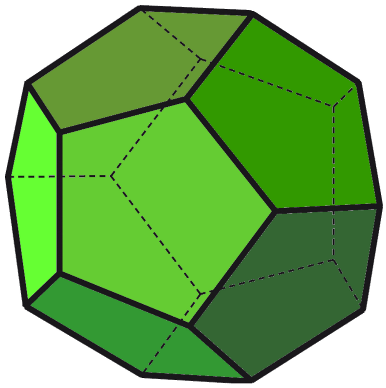
Task 4. Which discrete time martingales $\{M_n\}_{n=0}^\infty$ with $\mathbf{E}\{M_n^2\} < \infty$ for all n are also WSS processes (so that $\mathbf{E}\{M_m\}$ and $\mathbf{E}\{M_m M_{m+n}\}$ do not depend on m)? [HINT: Show that $\mathbf{E}\{M_n^2\} = \mathbf{E}\{(M_n - M_m)^2\} + \mathbf{E}\{M_m^2\}$ for $0 \leq m \leq n$.] **(5 points)**

Task 5. You are at liberty to select any zero-mean WSS process $\{X(t)\}_{t\in\mathbb{R}}$ you want as insignal to an LTI system with impulse response $h(t) = \sin(t)/(\pi\, t)$ for $t \in \mathbb{R}$. What are the restrictions on the possible outsignals $\{Y(t)\}_{t\in\mathbb{R}}$ from the LTI system?

(5 points)

TURN PAGE! ↷

Task 6. A regular dodecahedron is one of the five Platonic solids: It has 12 pentagonal (swedish: femsidiga) faces with three of them meeting in each of 20 vertices, see figure:



Consider a continuous random walk on the 20 vertices of a dodecahedron that spends an exponentially distributed time with mean 1/3 at each visit of a vertex after which one of the three neighbour vertices is selected as the next state of the random walk with equal probabilities 1/3. Consider the random time T it takes the random walk to move from its initial state/vertex to the state/vertex farthest away, i.e., five edges away. Find a linear system of equations that determines the characteristic function $\Psi_T(\omega)$ of T. (The equation system need not be solved - that is better done by Mathematica etc.)

(5 points)

MSG800/MVE170 Solutions to re-exam 7 April 2021

Task 1. $\mathbf{P}\{X(S)Y(T)>x\} = \sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\mathbf{P}\{N(0,k^2)>x\}\,\mathrm{e}^{-s}\mathrm{e}^{-t}\,\mathbf{P}\{X(s)=k\}\,dsdt = \sum_{k=1}^{\infty}\int_0^{\infty}(1-\Phi(x/k))\,\mathrm{e}^{-s}\,\frac{s^k}{k!}\,\mathrm{e}^{-s}\,ds = \sum_{k=1}^{\infty}2^{-(k+1)}\,(1-\Phi(x/k))$.

Task 2. We use that the sought after expectation is two times the expectation $\mathbf{E}\{\min\{t>0 : N(t)=3\}\}$ which we simulate as

```
In[1]:= Reps = 1000000;
In[2]:= For[i=1; time=0, i<=Reps, i++, Nt=0;
          While[Nt<3,
            If[Nt==0, Nt=1;
               time=time+Random[ExponentialDistribution[1]],
            If[Nt==1, If[Random[]<1/2, Nt=0, Nt=2];
               time=time+Random[ExponentialDistribution[2]],
            If[Nt==2, If[Random[]<1/2, Nt=1, Nt=3];
               time=time+Random[ExponentialDistribution[2]]]]]]];
In[3]:= 2*time/Reps
Out[3]= 11.9954
```

Task 3. Let P_i be the probability that Kal wins when he starts gaming with i trillion SEK for $i=1,2,3$. Then we have

$$P_1 = p\cdot P_2, \qquad P_2 = p\cdot P_3 + (1-p)\cdot P_1 \qquad \text{and} \qquad P_3 = p + (1-p)\cdot P_2.$$

This system of equations we solve to get $P_2 = p^2/(1-2p+2p^2)$.

Task 4. As $\mathbf{E}\{M_n^2\}-\mathbf{E}\{(M_n-M_m)^2\}-\mathbf{E}\{M_m^2\} = 2\,\mathbf{E}\{(M_n-M_m)M_m\} = 2\,\mathbf{E}\{\mathbf{E}\{(M_n-M_m)M_m|F_m\}\} = 2\,\mathbf{E}\{M_m\mathbf{E}\{M_n-M_m|F_m\}\} = 0$ for $0\le m\le n$ we must have $\mathbf{E}\{(M_n-M_m)^2\} = 0$ to make $\mathbf{E}\{M_m^2\}$ not depend on m implying that $M_n = M_0$ for all n. On the other hand $M_n = M_0$ is both a martingale and WSS so that is the answer.

Task 5. As the insignal is zero-mean $\mu_X = 0$, so must be the outsignal $\mu_Y = 0$. As for the autocorrelation function $R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{j\omega\tau}S_Y(\omega)\,d\omega$, since $H(\omega)=1$ for $|\omega|\le 1$ and 0 for $|\omega|>1$ we get $S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = 0$ for $|\omega|>1$ with no other restrictions on $S_Y(\omega)$ (except symmetry and positivity).

Task 6. With obvious notation we have

$$\begin{aligned}
\Psi_T(\omega) &= \tfrac{3}{3-j\omega}\,\Psi_{T_1}(\omega) \\
\Psi_{T_1}(\omega) &= \tfrac{3}{3-j\omega}\left[\tfrac{1}{3}\Psi_T(\omega)+\tfrac{2}{3}\Psi_{T_2}(\omega)\right] \\
\Psi_{T_2}(\omega) &= \tfrac{3}{3-j\omega}\left[\tfrac{1}{3}\Psi_{T_1}(\omega)+\tfrac{1}{3}\Psi_{T_2}(\omega)+\tfrac{1}{3}\Psi_{T_3}(\omega)\right] \\
\Psi_{T_3}(\omega) &= \tfrac{3}{3-j\omega}\left[\tfrac{1}{3}\Psi_{T_2}(\omega)+\tfrac{1}{3}\Psi_{T_3}(\omega)+\tfrac{1}{3}\Psi_{T_4}(\omega)\right] \\
\Psi_{T_4}(\omega) &= \tfrac{3}{3-j\omega}\left[\tfrac{2}{3}\Psi_{T_3}(\omega)+\tfrac{1}{3}\right]
\end{aligned} \; .$$