1 Analytical solution of computational task 1

Writing E_n for the expected value of the time it takes to reach the terminal state 2 starting in state $n=0,1,2$ we have the equations

$$E_0 = 1+(1/2)\cdot E_0+(1/3)\cdot E_1+(1/6)\cdot E_2, \quad E_1 = 1+(2/3)\cdot E_1+(1/3)\cdot E_2 \quad \text{and} \quad E_2=0$$

with solution $(E_0,E_1)=(4,3)$. Here E_0 is the expected value $E(T)$ asked for in the task.

On the left hand side of the two first equations we start at states 0 and 1, respectively, and on the right hand side we look one unit ahead in time (thus adding 1 to the expectation on the left hand side) and use the transition matrix P to calculate how likely it is that the journey to state 2 continues from the different possible states $(0,1,2)$ and $(1,2)$, respectively.

2 Quick proof of Stirling's formula for $n!$ as $n\to\infty$*

By the relation between faculties and the Gamma function, by Taylor expansion of $\ln(1+x)$ around $x=0$, and by recognition of a Gaussian PDF at the last step, we have, as $n\to\infty$,

$$\begin{aligned} n! = \int_0^\infty x^n\,\mathrm{e}^{-x}\,dx = \int_{-n}^\infty (y+n)^n\,\mathrm{e}^{-y-n}\,dy = n^n\mathrm{e}^{-n}\int_{-n}^\infty \mathrm{e}^{n\ln(1+y/n)-y}\,dy \equiv (\star) \\ = n^n\mathrm{e}^{-n}\int_{-n}^\infty \mathrm{e}^{-y^2/(2n)+o(y^2/n)}\,dy \sim \sqrt{2\pi n}\,n^n\mathrm{e}^{-n}. \end{aligned}$$

3 Justification of last row of above proof of Stirling's formula**

Clearly, by mentioned Taylor expansion, the expression $(\star)$ is greater or equal than

$$n^n\mathrm{e}^{-n}\int_{-n^{3/4}}^{n^{3/4}} \mathrm{e}^{-(1+\varepsilon)\,y^2/(2n)}\,dy \sim \sqrt{\frac{2\pi n}{1+\varepsilon}}\,n^n\mathrm{e}^{-n}$$

for any $\varepsilon>0$, for n large enough, where we can $\varepsilon\downarrow 0$ afterwards.

On the other hand, as $\ln(1+x)-x+x^2/2\leq 0$ for $x\leq 0$, the Taylor expansion shows that for each $\varepsilon>0$ there exists $\delta>0$ such that

$$n^n\mathrm{e}^{-n}\int_{-n}^{\delta n} \mathrm{e}^{n\ln(1+y/n)-y}\,dy \leq n^n\mathrm{e}^{-n}\int_{-n}^{\delta n} \mathrm{e}^{-(1-\varepsilon)\,y^2/(2n)}\,dy \sim \sqrt{\frac{2\pi n}{1-\varepsilon}}\,n^n\mathrm{e}^{-n}$$

for n large enough. Further, as $\ln(1+x)-(1-\delta/2)x\leq 0$ for $x\geq\delta$ and $\delta\in(0,1]$, we have

$$\int_{\delta n}^\infty \mathrm{e}^{n\ln(1+y/n)-y}\,dy = n\int_\delta^\infty \mathrm{e}^{n\ln(1+z)-nz}\,dz \leq n\int_\delta^\infty \mathrm{e}^{-(\delta/2)nz}\,dz \to 0 \quad \text{as } n\to\infty.$$