MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 10 January 2022 2–6 PM

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Consider a discrete time random process $\{X_n\}_{n=0}^{\infty}$ which takes the values 0, 1, or 2. Suppose that

$$\mathbf{P}\{X_{n+1}=j|X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\} = \begin{cases} P_{ij}^{I} & \text{when } n \text{ is even} \\ P_{ij}^{II} & \text{when } n \text{ is odd} \end{cases},$$

where P^I and P^{II} are stochastic matrixes (/OK Markov transition matrixes). Then $\{X_n\}_{n=0}^{\infty}$ is a non-time homogeneous Markov chain. How can one by enlarging the state space transform the process to a time homogeneous Markov chain? **(5 points)**

Task 2. Customers arrive at a bank as a Poisson process with rate $\lambda > 0$ per hour. Suppose two customers arrived during the first hour. What is the probability that

(a) both arrived during the first 20 minutes (=1/3 hour)? **(2,5 points)**

(b) at least one arrived during the first 20 minutes (=1/3 hour)? **(2,5 points)**

Task 3. Calculate $\mathbf{E}\{X(t_1)X(t_2)X(t_3)\}$ when $\{X(t)\}_{t\geq 0}$ is a Wiener process and $t_1, t_2, t_3 \geq 0$. **(5 points)**

Task 4. Machines in a factory break down at an exponential rate of six per hour. There is a single repairman who fixes machines at an exponential rate of eight per hour. The cost incurred in lost production when machines are out of service is $1000 per hour per machine. Find the average cost per hour incurred due to failed machines. **(5 points)**

Task 5. A single repairperson looks after two machines 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate λ_i, $i = 1, 2$. When machine i fails it requires an exponentially distributed time of work with rate μ_i, $i = 1, 2$, to complete its repair. The repairperson will always service machine 1 when it is down. So if machine 1 fails while 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start work on machine 1 (which destroy/make undone preceeding work spent on machine 2). Write up equations (that need not be solved) determining the proportion of time machine 2 is down. **(5 points)**

Task 6. Calculate $\mathbf{E}\{T\}$ for $T = \min\{t \geq 0 : X(t) = 2-4t\}$ and $\{X(t)\}_{t\geq 0}$ a Wiener process. HINT: The optional stopping theorem is valid also for continuous time martingales and $\mathbf{E}\{T\} \leq \sum_{n=0}^{\infty} \mathbf{P}\{T \geq n\}$ for any random variable $T \geq 0$. **(5 points)**

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Task 1. Let the state space be $E=\{0,1,2,\bar{0},\bar{1},\bar{2}\}$, where state i ($\bar{i}$) signifies that the present value is i and the present day is even (odd).

Task 2. **(a)** $\mathbf{P}\{X(1/3)=2|X(1)=2\} = \frac{\mathbf{P}\{X(1/3)=2,\, X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}}$
$= \frac{\mathbf{P}\{X(1/3)=2\}\,\mathbf{P}\{X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}} = \frac{((\lambda/3)^2/2!)\,\mathrm{e}^{-\lambda/3}\,((2\lambda/3)^0/0!)\,\mathrm{e}^{-2\lambda/3}}{(\lambda^2/2!)\,\mathrm{e}^{-\lambda}} = \frac{1}{9}$

(b) $\mathbf{P}\{X(1/3)\geq 1|X(1)=2\} = \frac{\mathbf{P}\{X(1/3)=1,\, X(1)-X(1/3)=1\}}{\mathbf{P}\{X(1)=2\}} + \frac{\mathbf{P}\{X(1/3)=2,\, X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}}$
$= \frac{\mathbf{P}\{X(1/3)=1\}\,\mathbf{P}\{X(1)-X(1/3)=1\}}{\mathbf{P}\{X(1)=2\}} + \frac{\mathbf{P}\{X(1/3)=2\}\,\mathbf{P}\{X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}} = \ldots = \frac{4}{9}+\frac{1}{9} = \frac{5}{9}$

Task 3. Suppose without loss that $t_3\geq t_2\geq t_1\geq 0$ and note that $\mathbf{E}\{X(t_1)X(t_2)X(t_3)\}$ $=\mathbf{E}\{(X(t_3)-X(t_2))X(t_2)X(t_1)\}+\mathbf{E}\{X(t_2)^2X(t_1)\} = 0+\mathbf{E}\{(X(t_2)-X(t_1))^2X(t_1)\}+$ $2\,\mathbf{E}\{X(t_2)X(t_1)^2\}-\mathbf{E}\{X(t_1)^3\} = 0+2\,\mathbf{E}\{(X(t_2)–X(t_1))X(t_1)^2\}+\mathbf{E}\{X(t_1)^3\}+0 = 0$ because of independent zero-mean increments and since $f_{X(t)}(x)$ is even so that $x^3 f_{X(t)}(x)$ is uneven making $\mathbf{E}\{X(t)^3\}=0$.

Task 4. This problem can be modeled by an M/M/1 queue with $\lambda=6$ and $\mu=8$. The average cost rate will be \$1000 per hour per machine times average number of broken machines $L=\frac{\lambda}{\mu-\lambda}=3$ so the answer is \$3000/hour.

Task 5. There are four states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and state 3 that both machines are down. The balance equations are as follows:

$$\begin{cases} (\lambda_1+\lambda_2)\,P_0 = \mu_1 P_1+\mu_2 P_2 \\ (\mu_1+\lambda_2)\,P_1 = \lambda_1 P_0 \\ (\lambda_1+\mu_2)\,P_2 = \lambda_2 P_0+\mu_1 P_3 \\ \mu_1 P_3 = \lambda_2 P_1+\lambda_1 P_2 \\ P_0+P_1+P_2+P_3 = 1 \end{cases}.$$

The proportion of time machine 2 is down is $P_2+P_3=\ldots$.

Task 6. As $\mathbf{P}\{T\geq n\}\leq \mathbf{P}\{X(n)\leq 2-4n\}$ which goes to zero very rapidly as $n\to\infty$ since $X(n)$ is $\mathrm{N}(0,n)$, we have $\mathbf{E}\{T\}\leq\sum_{n=0}^{\infty}\mathbf{P}\{T\geq n\}<\infty$ so that $\mathbf{E}\{|X(T)|\} = \mathbf{E}\{|2-4T|\}\leq 2+4\,\mathbf{E}\{T\}<\infty$ and (by Cauchy-Schwarz inequality) $\mathbf{E}\{|X(n)|I_{\{T>n\}}\}\leq (\mathbf{E}\{X(n)^2\})^{1/2}\,(\mathbf{P}\{T>n\})^{1/2}\to 0$ as $n\to\infty$. Hence the optional stopping theorem applies and $2-4\,\mathbf{E}\{T\}=\mathbf{E}\{X(T)\}=\mathbf{E}\{X(0)\}=0$ giving $\mathbf{E}\{T\}=1/2$.