MVE171 Basic Stochastic Processes and Financial Applications Written exam 28 August 2018 2–5 PM

TEACHER: Patrik Albin. JOUR: Jonatan Kallus 031 7725325.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 8 points for grade 3, 12 points for grade 4 and 16 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. A pair of WSS random processes $X(t)$ and $Y(t)$ are called jointly WSS if $\mathbf{E}\{X(t)Y(t+\tau)\}$ do no depend on t (but on τ only). Find two WSS random processes $X(t)$ and $Y(t)$ that are not jointly WSS. **(5 points)**

Task 2. A discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ has state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0,1,2\}, \qquad \mathbf{p}(0) = [1/6 \;\; 1/3 \;\; 1/2] \qquad \text{and} \qquad P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 2/3 & 1/6 & 1/6 \end{bmatrix},$$

respectively. Find the expected value $\mathbf{E}\{T\}$ of the time $T = \min\{n \geq 1 : X(n) \neq X(0)\}$ the chain spends in it first state. [HINT: For a random variable ξ with probability mass function $\mathbf{P}\{\xi = k\} = (1-p)^{k-1}p$ for $k = 1,2,3,\ldots$ it holds that $\mathbf{E}\{\xi\} = 1/p$.].

(5 points)

Task 3. A zero-mean WSS random signal (process) $\{X(t)\}_{t\in\mathbb{R}}$ with power spectral density $S_X(\omega)$ is transmitted on a noisy channel where an independent zero-mean random noise process $\{N(t)\}_{t\in\mathbb{R}}$ with power spectral density $S_N(\omega)$ is added to the signal so that the recived signal (process) is $Y(t) = X(t)+N(t)$. In an attempt to approximately reconstruct the originally transmitted signal $X(t)$ the recived signal $Y(t)$ is processed as insignal to an LTI system with frequency response $H(\omega) = S_X(\omega)/(S_X(\omega)+S_N(\omega))$ and output $Z(t)$. Show that the mean-square reconstruction error $\mathbf{E}\{(Z(t)-X(t))^2\}$ is given by $\int_{-\infty}^{\infty} S_X(\omega)S_N(\omega)/(S_X(\omega)+S_N(\omega))\,d\omega$. **(5 points)**

Task 4. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process [that is, a random process with stationary and independent zero-mean normal distributed increments and $W(0) = 0$] such that $\mathbf{E}\{W(1)^2\} = 1$. Find a function $f : [0,\infty) \to \mathbb{R}$ such that $\{W(t)^3 + f(t)W(t)\}_{t\geq 0}$ is a martingale with respect to the filtration containing all information of the history of the process $F_s = \sigma(W(r) : r \in [0,s])$. **(5 points)**

MVE171 Solutions to written exam 28 August 2018

Task 1. It is enough to put $Y(t)=X(-t)$ to arrive at $\mathbf{E}\{X(t)Y(t+\tau)\}=\mathbf{E}\{X(t)X(-t-\tau)\}=R_X(2t+\tau)$ which will depend on t for all non-degenerate WSS processes $X(t)$.

Task 2. $E[T]=(1/6)\cdot(1/(2/3))+(1/3)\cdot(1/(3/4))+(1/2)\cdot(1/(5/6))$.

Task 3. Writing $h(t)$ for the impulse response corresponding to the frequency response $H(\omega)$ [noting that $H(\omega)$ is real and positive] we have $\mathbf{E}\{(Z(t)-X(t))^2\}=\mathbf{E}\{((h\star X)(t)+(h\star N)(t)-X(t))^2\} = \mathbf{E}\{(h\star X)(t)^2+(h\star N)(t)^2+X(t)^2+2\,(h\star X)(t)\,(h\star N)(t)-2\,(h\star X)(t)\,X(t)-2\,(h\star N)(t)\,X(t)\} = \int_{-\infty}^{\infty} H(\omega)^2 S_X(\omega)\,d\omega + \int_{-\infty}^{\infty} H(\omega)^2 S_N(\omega)\,d\omega + \int_{-\infty}^{\infty} S_X(\omega)\,d\omega + 0 - 2\int_{-\infty}^{\infty} H(\omega)S_X(\omega)\,d\omega - 0 = \int_{-\infty}^{\infty} S_X(\omega)S_N(\omega)/(S_X(\omega)+S_N(\omega))\,d\omega$.

Task 4. As $\mathbf{E}\{W(t)^3|F_s\}=\mathbf{E}\{(W(t)-W(s)+W(s))^3|F_s\}=\mathbf{E}\{(W(t)-W(s))^3|F_s\}+3\,\mathbf{E}\{(W(t)-W(s))^2W(s)\,|\,F_s\}+3\,\mathbf{E}\{(W(t)-W(s))\,W(s)^2|\,F_s\}+\mathbf{E}\{W(s)^3|F_s\}=\mathbf{E}\{(W(t)-W(s))^3\}+3\,W(s)\,\mathbf{E}\{(W(t)-W(s))^2\}+3\,W(s)^2\,\mathbf{E}\{W(t)-W(s)\}+W(s)^3=0+3\,W(s)\times(t-s)+0+W(s)^3$ for $0\le s\le t$ so we must take $f(t)=-3t$.