MVE171 Basic Stochastic Processes and Financial Applications Written exam Saturday 8 December 2018 8.30–11.30

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8 points for grade 3, 12 points for grade 4 and 16 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a WSS zero-mean Gaussian random process with autocorrelation function $R_X(\tau) = \mathrm{e}^{-|\tau|}$. Find the probability that $\mathbf{P}\{X(3) \geq X(1)+X(2)\}$.

(5 points)

Task 2. A discrete time Markov chain $\{X(n)\}_{n=0}^{\infty}$ has state space E, initial distribution $\boldsymbol{p}(0)$ and transition probability matrix P given by

$$E = \{0, 1, 2\}, \qquad \boldsymbol{p}(0) = [1/3 \;\; 1/3 \;\; 1/3] \qquad \text{and} \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix},$$

respectively. Find the expected value $\mathbf{E}\{T_2\}$ of the random variable $T_2 = \min\{n \geq 0 : X(n) = 2\}$. [HINT: For a random variable ξ with probability mass function $\mathbf{P}\{\xi = k\} = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$ it holds that $\mathbf{E}\{\xi\} = 1/p$.] **(5 points)**

Task 3. Let $X_n = \sum_{k=1}^{n} Y_k$ for $n = 1, 2\ldots$ and $X_0 = 0$ where $Y_1, Y_2, \ldots$ are independent zero-mean random variables with unit variance. Show that $\{X_n^2 - n\}_{n=0}^{\infty}$ is a martingale with respect to the filtration containing all information of the history of the process $F_m = \sigma(Y_0, \ldots, Y_m)$. **(5 points)**

Task 4. Consider an M/M/1/K queueing system with $\lambda = \mu = 1$. Calculate the average arrival rate of of customers that really enters the queueing system (and do not "bounce off" because system is full). **(5 points)**

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Task 1. $\mathbf{P}\{X(3) \geq X(1)+X(2)\} = \mathbf{P}\{X(3)-X(1)-X(2) \geq 0\} = \mathbf{P}\{\mathrm{N}(0,\sigma^2) \geq 0\} = 1/2$ as $\sigma^2 = 3 - 2\,\mathrm{e}^{-2} > 0$.

Task 2. $\mathbf{E}\{T_2\} = (1/3) \cdot 1 + (1/3) \cdot (1/(1/2)) + 1/3 \cdot 0 = 1$.

Task 3. $\mathbf{E}\{X_n^2 - n | F_m\} = \mathbf{E}\{(X_n - X_m)^2 | F_m\} + \mathbf{E}\{2\,(X_n - X_m)\,X_m | F_m\} + \mathbf{E}\{X_m^2 | F_m\} - n = \mathbf{E}\{(X_n - X_m^2\} + 2\,X_m\,\mathbf{E}\{X_n - X_m\} + X_m^2 - n = (n-m) + 0 + X_m^2 - n = X_m^2 - m$ for $0 \leq m < n$.

Task 4. As $p_n = 1/(K+1)$ for $n = 0, \ldots, K$ we have $\lambda_a = \lambda\,(1-p_K) = K/(K+1)$.