Projects in Financial Mathematics

Simone Calogero

Chalmers University of Technology

Contents

Chapter 0

Background

0.1 Finite probability theory

We begin by recalling a few results on finite probability spaces. For more details on this subject, see Chapter 5 in [2].

Let $\Omega = \{\omega_1, \dots, \omega_m\}$ be a sample space containing m elements. Let $p = (p_1, \dots, p_m)$ be a **probability vector**, i.e.,

$$0 < p_i < 1, \text{ for all } i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1.$$

We define $p_i = \mathbb{P}(\{\omega_i\})$ to be the probability of the event $\{\omega_i\}$. If $A \subseteq \Omega$ is a non-empty event, we define the probability of A as

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} p_i = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

Moreover $\mathbb{P}(\emptyset) = 0$. The pair $(\Omega, \mathbb{P})$ is called a **finite probability space**. For example, given $p \in (0, 1)$, the probability space

$$\Omega_N = \{H, T\}^N, \quad \mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}$$

is called the N**-coin toss probability space**. Here $N_H(\omega)$ is the number of heads in the toss $\omega \in \Omega_N$ and $N_T(\omega) = N - N_H(\omega)$ is the number of tails. In this probability space, tosses are independent and each toss has the same probability p to result in a head.

A **random variable** is a function $X : \Omega \to \mathbb{R}$. Y is said to be X**-measurable** if there exists a function g such that $Y = g(X)$. Two random variables X, Y are **independent** if $\mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J)$ for every $I \subseteq \mathrm{Im}(X)$ and $J \subseteq \mathrm{Im}(Y)$, where $\mathrm{Im}(X) = \{y \in \mathbb{R} : y = X(\omega) \text{ for some } \omega \in \Omega\}$ is the image of X.

The **expectation** of X is denoted by $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\{\omega\}).$$

and satisfies the properties in the following theorem.

Theorem 0.1. *Let X, Y be random variables, $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$. The following holds:*

1. $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$ *(linearity).*
2. *If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.*
3. *If X, Y are independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*
4. *If $Y = g(X)$, i.e., if Y is X-measurable, then*

$$\mathbb{E}[g(X)] = \sum_{x\in\mathrm{Im}(X)} g(x)f_X(x), \tag{1}$$

where $f_X(x) = \mathbb{P}(X = x)$ is the **probability distribution** *of the random variable X.*

For instance in the N-coin toss probability space consider a random variable X which is measurable with respect to N_H, i.e., $X(\omega) = g(N_H(\omega))$. Then

$$\begin{aligned}\mathbb{E}_p[X] &= \sum_{\omega\in\Omega_N} X(\omega)\mathbb{P}(\{\omega\}) = \sum_{\omega\in\Omega_N} g(N_T(\omega))p^{N_H(\omega)}(1-p)^{N_T(\omega)} \\ &= \sum_{k=0}^{N} \binom{N}{k} g(k)p^k(1-p)^k, \end{aligned} \tag{2}$$

where we used that the are $\binom{N}{k}$ N-tosses such that $N_T(\omega) = k$.
The quantity

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

is called **variance** of the random variable X. The quantity

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

is called **covariance** of the random variables X, Y. We have the identities

$$\mathrm{Var}[X] = \mathrm{Cov}[X, X], \quad \mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X, Y].$$

If $\mathrm{Var}[X], \mathrm{Var}[Y]$ are both positive (i.e., if X, Y are not deterministic constants), the quantity

$$\mathrm{Corr}[X, Y] = \frac{\mathrm{Cov[X, Y]}}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \in [-1, 1]$$

is called **correlation** of X, Y. If $\mathrm{Corr}[X, Y] = 0$, the random variables X, Y are said to be **uncorrelated**. It follows by Theorem 0.1(3) that X, Y independent $\Rightarrow X, Y$ uncorrelated (while the opposite is in general not true).

The **conditional expectation** of X given Y is denoted by $\mathbb{E}[X|Y]$:

$$\mathbb{E}[X|Y](\omega) = \sum_{x \in \mathrm{Im}(X)} \mathbb{P}(X = x | Y = Y(\omega)) x,$$

where $\mathbb{P}(A|B) = \mathbb{P}(B)^{-1}\mathbb{P}(A \cap B)$ is the conditional probability of the event A given the event B. The conditional expectation is a Y-measurable random variable and satisfies the following properties.

Theorem 0.2. *Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and $\alpha, \beta \in \mathbb{R}$. Then*

1. $\mathbb{E}[\alpha X + \beta Y|Z] = \alpha\mathbb{E}[X|Z] + \beta\mathbb{E}[Y|Z]$ *(linearity).*
2. *If X is independent of Y, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$.*
3. *If X is Y-measurable, then $\mathbb{E}[X|Y] = X$.*
4. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
5. *If X is Z-measurable, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$.*
6. *If Z is Y-measurable then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$.*

These properties remain true if the conditional expectation is taken with respect to several random variables.

A **discrete stochastic process** is a (possibly finite) sequence $\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n\in\mathbb{N}}$ of random variables. We refer to the index n in X_n as **time step**. If the discrete stochastic process is finite, i.e., if it runs only for a finite number $N \geq 1$ of time steps, we shall denote it by $\{X_n\}_{n=0,\dots,N}$ and call it a **N-period process**. At each time step, a discrete stochastic process on a finite probability space is a random variable with finitely many possible values. More precisely, for all $n = 0, 1, 2, \dots$, the value x_n of X_n satisfies $x_n \in \mathrm{Im}(X_n)$. We call x_n an **admissible state** of the stochastic process. Note that x_n is an admissible state if and only if $\mathbb{P}(X_n = x_n) > 0$.

A stochastic process $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **measurable** with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ there exists a function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $Y_n = g_n(X_0, X_2, \dots, X_n)$. If $Y_n = h_n(X_0, \dots, X_{n-1})$ for some function $h_n : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, then $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **predictable** from the process $\{X_n\}_{n\in\mathbb{N}}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **martingale** if

$$\mathbb{E}[X_{n+1}|X_1, X_2, \dots X_n] = X_n, \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

The interpretation is the following: The variables $X_0, X_1, \dots X_n$ contains the information obtained by "looking" at the stochastic process up to the time step n. For a martingale process, this information is not enough to estimate whether, in the next step, the process will raise or fall. Martingales have constant expectation, i.e., $\mathbb{E}[X_n] = \mathbb{E}[X_0]$, for all $n \in \mathbb{N}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **Markov chain** if it satisfies the **Markov property**:

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1}|X_1 = x_1, X_2 = x_2, \dots, X_n = x_n), \qquad (4)$$

for all $n \in \mathbb{N}$ and for all admissible states $x_0 \in \mathrm{Im}(X_0), \dots, x_{n+1} \in \mathrm{Im}(X_{n+1})$ such that $\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots X_n = x_n)$ is positive[1]. The interpretation is the following: If $\{X_n\}_{n\in\mathbb{N}}$ is a Markov process, then the probability of transition from the state x_n to the state x_{n+1} does not depend on the states occupied by the process before time n. Thus Markov processes are "memoryless": at each time step they "forget" what they did earlier.

Remark 0.1. If $\{X_n\}_{n\in\mathbb{N}}$ is a Markov process and $\{Y_n\}_{n\in\mathbb{N}}$ is measurable with respect to $\{X_n\}_{n\in\mathbb{N}}$, then the Markov property (4) implies

$$\mathbb{E}[Y_n|X_{n-1}] = \mathbb{E}[Y_n|X_0, \dots X_{n-1}]$$

The left hand side of (4) is called the **transition probability** from the state x_n to the state x_{n+1} and is denoted also as $\mathbb{P}(x_n \to x_{n+1})$. If $\mathbb{P}(x_n \to x_{n+1})$ is independent of $n = 1, 2, \dots$, the Markov process is said to be **time homogeneous**.

Note that both the Markov property and the martingale property depend on the probability measure, i.e., a stochastic process can be a martingale and/or a Markov process in one probability $\mathbb{P}$ and neither of them in another probability $\mathbb{P}'$.

Example: Random Walk. Consider the following stochastic process $\{X_n\}_{n=1,\dots,N}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$$\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N, \quad X_n(\omega) = \begin{cases} 1 & \text{if } \gamma_n = H \\ -1 & \text{if } \gamma_n = T \end{cases}.$$

The random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d**), namely

$$\mathbb{P}_p(X_n = 1) = p, \quad \mathbb{P}_p(X_n = -1) = 1 - p, \quad \text{for all } n = 1, \dots, N.$$

Hence

$$\mathbb{E}[X_n] = 2p - 1, \quad \mathrm{Var}[X_n] = 4p(1-p), \quad \text{for all } n = 1, \dots, N.$$

Now, for $n = 1, \dots, N$, let

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i.$$

[1] That is to say, there must be a path of the stochastic process that connects the states $x_0, \dots, x_n$.

The stochastic process $\{M_n\}_{n=0,\dots,N}$ is measurable (but not predictable) with respect to the process $\{X_n\}_{n=1,\dots,N}$ and is called **(N-period) random walk**. It satisfies

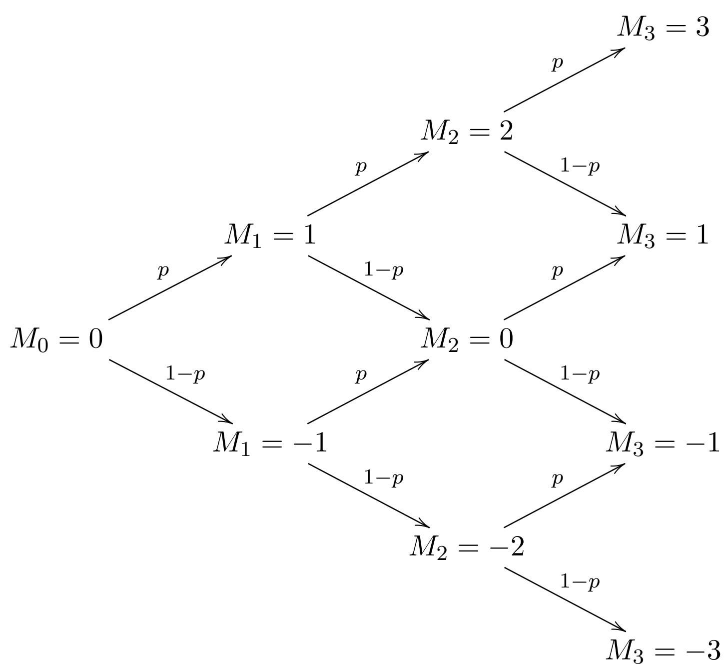
$$\mathbb{E}[M_n] = n(2p-1), \quad \text{for all } n = 0, \dots, N.$$

Moreover, since it is the sum of independent random variables, the random walk has variance given by

$$\mathrm{Var}[M_0] = 0, \quad \mathrm{Var}[M_n] = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}[X_i] = 4np(1-p).$$

When $p = 1/2$, the random walk is said to be **symmetric**. In this case $\{M_n\}_{n=0,\dots,N}$ satisfies $\mathbb{E}[M_n] = 0$ and $\mathrm{Var}[M_n] = n$, $n = 0, \dots, N$. When $p \neq 1/2$, $\{M_n\}_{n=0,\dots,N}$ is called an **asymmetric** random walk, or a random walk with **drift**.

If $M_n = k$ then M_{n+1} is either $k+1$ (with probability p), or $k-1$ (with probability $1-p$). Hence we can represent the paths of the random walk by using a binomial tree, as in the following example for $N = 3$:



By inspection we see that the admissible states of the symmetric random walk at the step n are given by

$$\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\}.$$

Let $m_0 = 0$, $m_1 \in \{-1, 1\} = \mathrm{Im}(M_1)$, $\dots$, $m_N \in \{-N, -N+2, \dots, N-2, N\} = \mathrm{Im}(M_N)$ be the admissible states at each time step. From the binomial tree of the process it is clear that there exists a path connecting $m_0, m_1, \dots, m_N$ if and only if $m_n = m_{n-1} \pm 1$, for all

$n = 1, \ldots, N$, and we have

$$\begin{aligned}\mathbb{P}(M_n = m_n | M_1 = m_1, \ldots, M_{n-1} = m_{n-1}) &= \mathbb{P}(M_n = m_n | M_{n-1} = m_{n-1}) \\ &= \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \end{cases}\end{aligned}$$

Hence the random walk is an example of time homogeneous Markov chain.

Next we show that the *symmetric* random walk is a martingale. In fact, using the linearity of the conditional expectation we have

$$\begin{aligned}\mathbb{E}[M_n | M_1, \ldots, M_{n-1}] &= \mathbb{E}[M_{n-1} + X_n | M_1, \ldots, M_{n-1}] \\ &= \mathbb{E}[M_{n-1} | M_1, \ldots, M_{n-1}] + \mathbb{E}[X_n | M_1, \ldots, M_{n-1}].\end{aligned}$$

As M_{n-1} is measurable with respect to $M_1, \ldots, M_{n-1}$, then $\mathbb{E}[M_{n-1}|M_1, \ldots, M_{n-1}] = M_{n-1}$, see Theorem 0.2(3). Moreover, as X_n is independent of $M_1, \ldots, M_{n-1}$, Theorem 0.2(2) gives $\mathbb{E}[X_n|M_1, \ldots, M_{n-1}] = \mathbb{E}[X_n] = 0$. It follows that $\mathbb{E}[M_n|M_1, \ldots, M_{n-1}] = M_{n-1}$, i.e., the symmetric random walk is a martingale. However the asymmetric random walk ($p \neq 1/2$) is *not* a martingale, as it follows by the fact that its expectation $\mathbb{E}[M_n] = n(2p - 1)$ is not constant.

Generalized random walk. A random walk is any discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ which satisfies the following properties:

- $\mathrm{Im}(M_n) = \{-n, -n + 2, -n + 4, \ldots, n - 2, n\}$, for all $n = 0, 1, \ldots$
- $\{M_n\}_{n\in\mathbb{N}}$ is a time-homogeneous Markov chain
- There exists $p \in (0, 1)$ such that for $(m_{n-1}, m_n) \in \mathrm{Im}(M_{n-1}) \times \mathrm{Im}(M_n)$, the transition probability $\mathbb{P}(m_{n-1} \to m_n)$ is given by

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}$$

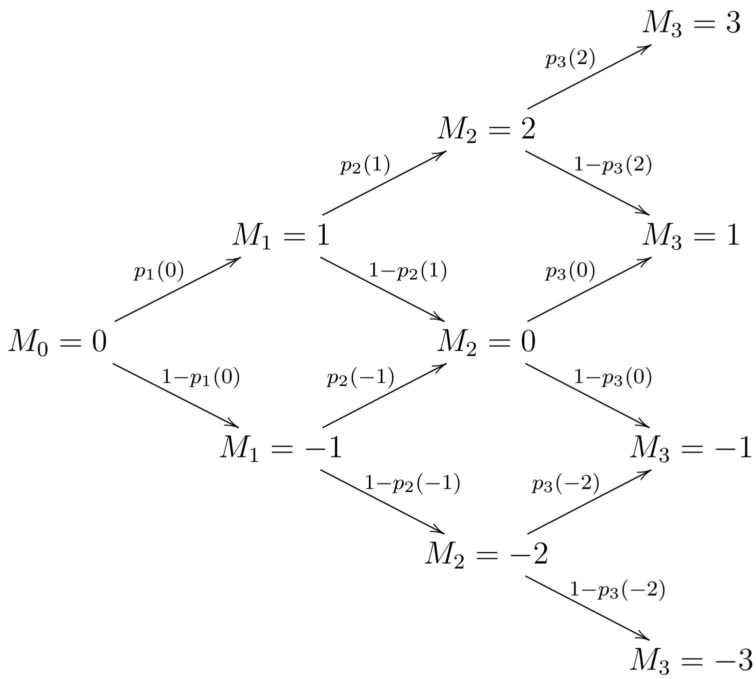
We may generalize this definition by relaxing the second and third properties as follows.

Definition 0.1. *A discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ on a finite probability space is called a* **generalized random walk** *if it satisfies the following properties:*

1. $\mathrm{Im}(M_n) = \{-n, -n + 2, -n + 4, \ldots, n - 2, n\}$, *for all* $n = 0, 1, \ldots$
2. $\{M_n\}_{n\in\mathbb{N}}$ *is a Markov chain*
3. *For all* $n = 1, 2, \ldots$ *there exist* $p_n : \mathrm{Im}(M_{n-1}) \to (0, 1)$ *such that*

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p_n(m_{n-1}) & \text{if } m_n = m_{n-1} + 1 \\ 1 - p_n(m_{n-1}) & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}$$

The binomial tree of a generalized random walk will be written as in the following example:



Note that when $p_n \equiv p$ for all $n = 1, 2, \dots$, the generalized random walk becomes the standard random walk considered before.

For later purpose we give below a formula to compute the probability that the generalized random walk follows a given path. It is clear that any path in the N-period random walk is uniquely identified by a vector $x \in \{-1, 1\}^N$, i.e., a N-dimensional vector where each component is either -1 or 1. More precisely, the path of the random walk corresponding to $x \in \{-1, 1\}^N$ it the unique path satisfying $M_0 = 0$ and $M_i = M_{i-1} + x_i$, $i = 1, \dots, N$.

Theorem 0.3. *Let $x \in \{-1, 1\}^N$ and set $x_0 = 0$. The probability $\mathbb{P}(x)$ that the generalized random walk follows that path x is given by*

$$\mathbb{P}(x) = \prod_{k=1}^{N} \left[-\min(x_k, 0) + x_k p_k \left(\sum_{j=0}^{k-1} x_j \right) \right]. \tag{5}$$

The previous theorem can be easily proved by induction, but here we limit ourselves to consider one example of application of (5). In the 3-period model consider the path $x = (-1, -1, 1)$. Then according to the previous theorem

$$\begin{aligned} \mathbb{P}((-1,-1,1)) &= (-\min(-1,0) + (-1)p_1(0))(-\min(-1,0) + (-1)p_2(0-1)) \\ &\quad \times (-\min(1,0) + (1)p_2(0-1-1)) = (1-p_1(0))(1-p_2(-1))p_2(-2). \end{aligned}$$

That this formula is correct is easily seen in the binomial tree above.

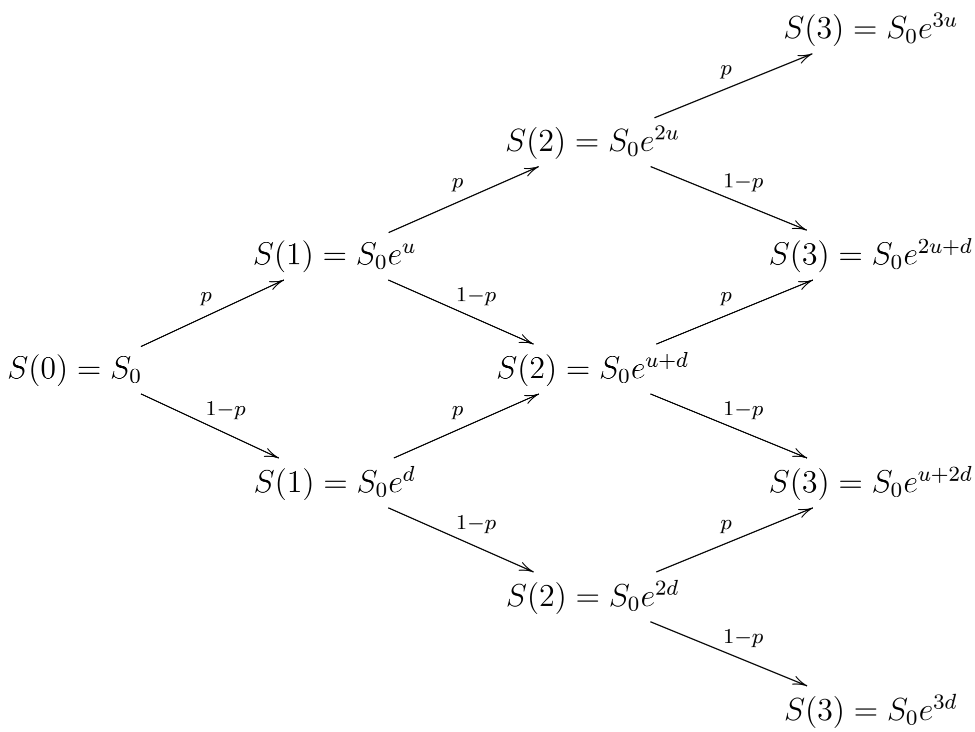
The generalized random walk will be used in the project in Chapter 1 to introduce and study the properties of a generalized binomial model in which the interest rate of the risk-free asset is a stochastic process. The standard binomial model, in which the risk-free rate is assumed to be constant, is review in the next section.

0.2 The binomial options pricing model

Given $0 < p < 1$, $S_0 > 0$ and $u > d$, the **binomial stock price** at time t is given by $S(0) = S_0$ and

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \qquad \text{for } t = 1, \dots, N. \tag{6}$$

If $S(t) = S(t-1)e^u$ we say that the stock price goes up at time t, while if $S(t) = S(t-1)e^d$ we say that it goes down at time t (although this terminology is strictly correct only when $u > 0$ and $d < 0$). For instance, for $N = 3$ the binomial stock can be represented as in the following recombining binomial tree:



The possible stock prices at time t belong to the set

$$\text{Im}(S(t)) = \{S_0 e^{N_u(t)u + (t - N_u(t))d}, \; N_u(t) = 0, \dots, t\},$$

where $N_u(t)$ is the number of times that the price goes up up to and including time t. It follows that there are $t + 1$ possible prices at time t and so the number of nodes in the binomial tree grows linearly in time.

The binomial stock price can be interpreted as a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. To see this, consider the following i.i.d. random variables

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \begin{cases} 1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H \\ -1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T \end{cases}, \quad t = 1, \dots, N. \tag{7}$$

We can rewrite (6) as $S(t) = S(t-1)\exp[(u+d)/2 + (u-d)X_t/2]$, which upon iteration leads to

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t\right], \quad M_t = X_1 + \dots + X_t, \ t = 1, \dots, N. \tag{8}$$

Hence $S(t) : \Omega_N \to \mathbb{R}$ and therefore $\{S(t)\}_{t=0,\dots,N}$ is a N-period stochastic process on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. In this context, $\mathbb{P}_p$ is called **physical** (or **real-world**) **probability measure**, to distinguish it from the martingale (or risk-neutral) probability introduced below. Letting $M_0 = 0$, we have that $\{M_t\}_{t=0,\dots,N}$ is a random walk (which is asymmetric for $p \neq 1/2$). It follows that $\{S(t)\}_{t=0,\dots,N}$ is measurable, but not predictable, with respect to $\{M_t\}_{t=0,\dots,N}$. For each $\omega \in \Omega_N$, the vector $(S(0), S(1,\omega), \dots, S(N,\omega))$ is called a **path** of the binomial stock price.

A **binomial market** is a market that consists of one stock with price given by (8), and a **risk-free asset** with value $B(t)$ at time $t = 1, \dots, N$. In the standard binomial model it is assumed that $B(t)$ is a deterministic function of time with constant interest rate, namely

$$r = \log B(t+1) - \log B(t), \quad \text{or} \quad R = \frac{B(t+1) - B(t)}{B(t)}.$$

It follows that the value of the risk-free asset at time t can be written in either of the two forms

$$B(t) = B_0 e^{rt}, \quad B(t) = (1+R)^t, \quad t = 1, \dots, N,$$

where B_0 is the initial value of the risk-free asset. We shall refer to R as the **discretely compounded risk-free rate** and to r as the **continuously compounded risk-free rate** (although the latter terminology is only strictly correct in the time continuum limit, i.e., when we let the length of the time step tends to zero). Note also that

$$r = \log(1+R). \tag{9}$$

As r and R are small, then $r \approx R$.

Remark 0.2. In [2] only the risk-free rate r was used. Here we introduced the discretely compounded risk-free rate R as well because it will be used in Chapter 1 to formulate a generalized binomial model with stochastic risk-free rate.

The quantity

$$S^*(t) = e^{-rt} S(t), \quad \text{or equivalently} \quad S^*(t) = \frac{S(t)}{(1+R)^t},$$

is called the **discounted price** of the stock (at time $t = 0$).

In the following we denote by $\mathbb{E}_p$ the (possibly conditional) expectation in the probability space $(\Omega_N, \mathbb{P}_p)$.

Theorem 0.4. *If $r \notin (d, u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d, u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where*

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

Proof. By definition, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale if and only if

$$\mathbb{E}_p[S^*(t)|S^*(0), \dots, S^*(t-1)] = S^*(t-1), \quad \text{for all } t = 1, \dots, N.$$

Taking the expectation conditional to $S^*(0), \dots, S^*(t-1)$ is clearly the same as taking the expectation conditional to $S(0), \dots, S(t-1)$, hence the above equation is equivalent to

$$\mathbb{E}_p[S(t)|S(0), \dots, S(t-1)] = e^r S(t-1), \quad \text{for all } t = 1, \dots, N, \tag{10}$$

where we canceled out a factor e^{-rt} in both sides of the equation. Moreover

$$\begin{aligned}\mathbb{E}_p[S(t)|S(0), \dots, S(t-1)] &= \mathbb{E}_p[\frac{S(t)}{S(t-1)} S(t-1)|S(0), \dots, S(t-1)] \\ &= S(t-1)\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0), \dots, S(t-1)],\end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditioning variables and thus it can be taken out from the conditional expectation (see property 5 in Theorem 0.2). As

$$S(t)/S(t-1) = \begin{cases} e^u & \text{with prob. } p \\ e^d & \text{with prob. } 1-p \end{cases}$$

is independent of $S(0), \dots, S(t-1)$, then by Theorem 0.2(2) we have

$$\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0), \dots, S(t-1)] = \mathbb{E}_p[\frac{S(t)}{S(t-1)}] = e^u p + e^d(1-p).$$

Hence (10) holds if and only if $e^u p + e^d(1-p) = e^r$. Solving in $p \in (0, 1)$ we find $p = q$ and the condition $0 < q < 1$ is then equivalent to $r \in (d, u)$. □

Due to Theorem 0.4, $\mathbb{P}_q$ is called **martingale probability measure**. Moreover, since martingales have constant expectation, then

$$\mathbb{E}_q[S(t)] = S_0 e^{rt}. \tag{11}$$

Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk-neutral probability**.

Self-financing portfolios

A **portfolio process** in a binomial market is a stochastic process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ such that, for $t = 1,\dots,N$, $(h_S(t), h_B(t))$ corresponds to the portfolio position (number of shares) on the stock and the risk-free asset held in the interval $(t-1, t]$. A positive number of shares corresponds to a long position on the asset, while a negative number of shares corresponds to a short position. As portfolio positions held for one instant of time only are meaningless, we use the convention $h_S(0) = h_S(1), h_B(0) = h_B(1)$, that is to say, $h_S(1), h_B(1)$ is the portfolio position in the *closed* interval $[0, 1]$. We always assume that the portfolio process is predictable from $\{S(t)\}_{t=0,\dots,N}$, i.e., there exists functions $H_t : (0,\infty)^t \to \mathbb{R}^2$ such that $(h_S(t), h_B(t)) = H_t(S(0),\dots,S(t-1))$. Thus the decision on which position the investor should take in the interval $(t-1, t]$ depends only on the information available at time $t-1$. The **value** of the portfolio process is the stochastic process $\{V(t)\}_{t=0,\dots,N}$ given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t), \quad t = 0,\dots,N. \tag{12}$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ is said to be **self-financing** if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1), \quad t = 1,\dots,N, \tag{13}$$

while it is said to generate the **cash flow** $C(t-1)$ if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1) + C(t-1), \quad t = 1,\dots,N. \tag{14}$$

Recall that $C(t) > 0$ corresponds to cash withdrawn from the portfolio at time t while $C(t) < 0$ corresponds to cash added to the portfolio at time t. The self-financing property means that no cash is ever added or withdrawn from the portfolio.

Theorem 0.5. *Let $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ be a self-financing predictable portfolio process with value $\{V(t)\}_{t=0,\dots,N}$. Then the discounted portfolio value $\{V^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability measure. Moreover the following identity holds:*

$$V^*(t) = \mathbb{E}_q[V^*(N)|S(0),\dots,S(t)], \quad t = 0,\dots,N. \tag{15}$$

Proof. The martingale claim is

$$\mathbb{E}_q[V^*(t)|V^*(0),\dots,V^*(t-1)] = V^*(t-1).$$

We now show that this follows by

$$\mathbb{E}_q[V^*(t)|S(0),\dots,S(t-1)] = V^*(t-1). \tag{16}$$

In fact, computing the expectation of (16) conditional to $V^*(0),\dots,V^*(t-1)$, we obtain

$$\begin{aligned} V^*(t-1) &= \mathbb{E}_q[V^*(t-1)|V^*(0),\dots,V^*(t-1)] \\ &= \mathbb{E}_q\big[\mathbb{E}_q[V^*(t)|S(0),\dots,S(t-1)]|V^*(0),\dots,V^*(t-1)\big] \\ &= \mathbb{E}_q[V^*(t)|V^*(0),\dots,V^*(t-1)], \end{aligned}$$

where we have used property 3 of Theorem 0.2 in the first equality and property 6 in the last equality. The latter is possible because $V^*(t)$ is measurable with respect to $S(0), \dots, S(t)$. Now we claim that (16) also implies the formula (15). We argue by backward induction. Letting $t = N$ in (16) we see that (15) holds at $t = N - 1$. Assume now that (15) holds at time $t + 1$, i.e.,

$$V^*(t+1) = \mathbb{E}_q[V^*(N)|S(0), \dots, S(t+1)].$$

Taking the expectation conditional to $S(0), \dots, S(t)$ we have, by (16),

$$\begin{aligned} V^*(t) &= \mathbb{E}_q[V^*(t+1)|S(0), \dots, S(t)] = \mathbb{E}_q\Big[\mathbb{E}_q[V^*(N)|S(0), \dots, S(t+1)]|S(0), \dots, S(t)\Big] \\ &= \mathbb{E}_q[V^*(N)|S(0), \dots, S(t)]. \end{aligned}$$

Hence (15) holds at time t and so (16) $\Rightarrow$ (15), as claimed. Finally we prove (16). As $B(t) = B(t-1)e^r$, (13) gives

$$h_B(t)B(t) = e^r V(t-1) - h_S(t)S(t-1)e^r.$$

Replacing in (12) we find

$$V(t) = e^r V(t-1) + h_S(t)[S(t) - S(t-1)e^r].$$

Taking the expectation conditional to $S(0), \dots, S(t-1)$ we obtain

$$\begin{aligned} \mathbb{E}_q[V(t)|S(0), \dots S(t-1)] &= e^r \mathbb{E}_q[V(t-1)|S(0), \dots, S(t-1)] \\ &\quad + \mathbb{E}_q[h_S(t)(S(t) - S(t-1)e^r)|S(0), \dots, S(t-1)]. \end{aligned} \tag{17}$$

As $V(t-1)$ and $h_S(t)$ are measurable with respect to the conditioning variables we have $\mathbb{E}_q[V(t-1)|S(0), \dots, S(t-1)] = V(t-1)$, as well as

$$\begin{aligned} &\mathbb{E}_q[h_S(t)(S(t) - S(t-1)e^r)|S(0), \dots, S(t-1)] \\ &= h_S(t)\mathbb{E}_q[S(t) - S(t-1)e^r|S(0), \dots, S(t-1)] \\ &= h_S(t)\Big(\mathbb{E}_q[S(t)|S(0), \dots, S(t-1)] - S(t-1)e^r\Big) = 0, \end{aligned}$$

where in the last step we used that $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability. Going back to (17) we obtain

$$\mathbb{E}_q[V(t)|S(0), \dots S(t-1)] = e^r V(t-1),$$

which is the same as (16). □

Arbitrage portfolios

A portfolio process $\{(h_S(t), h_B(t)\}_{t=0,\dots,N}$ invested in the binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;

2) $V(N, \omega) \geq 0$, for all $\omega \in \Omega_N$;

3) There exists $\omega_* \in \Omega_N$ such that $V(N, \omega_*) > 0$.

Theorem 0.6. *Assume $d < r < u$, i.e., assume the existence of a risk-neutral probability measure for the binomial market. Then the binomial market is free of self-financing arbitrages.*

Proof. Assume that $\{h_S(t), h_B(t)\}_{t=0,\dots,N}$ is a self-financing arbitrage portfolio process. Then $V(0) = V^*(0) = 0$ and since martingales have constant expectation then $\mathbb{E}_q[V^*(t)] = 0$, for all $t = 0, 1, \dots, N$. As $V(N) \geq 0$, then $V^*(N) \geq 0$ and Theorem 0.1(2) entails $V^*(N, \omega) = 0$ for any sample $\omega \in \Omega_N$. Hence $V(N, \omega) = 0$, for all $\omega \in \Omega_N$, contradicting the assumption that the portfolio is an arbitrage. □

Remark 0.3. As shown in [2], the existence of a risk-neutral probability measure in not only sufficient but also necessary for the absence of self-financing arbitrages in the binomial market. More precisely, if $r \notin (d, u)$ one can construct self-financing arbitrage portfolios in the market. Hence the binomial market is free of self-financing arbitrages if and only if it admits a risk-neutral probability measure. The latter result is valid for any discrete (or even continuum) market model and is known as the **first fundamental theorem of asset pricing**.

Risk neutral pricing formula for European derivatives in the binomial model

Let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European-style derivative with pay-off Y at maturity time $T = N$. This means that the derivative can only be exercised at time $t = N$. For standard European derivatives Y is a deterministic function of $S(N)$, while for non-standard derivatives Y is a deterministic function of $S(0), \dots, S(N)$. Let $\Pi_Y(t)$ be the binomial fair price of the derivative a time t. By definition, $\Pi_Y(t)$ equals the value $V(t)$ of self-financing, hedging portfolios. In particular, $\Pi_Y(t)$ is a random variable and so $\{\Pi_Y(t)\}_{t=0,\dots,N}$ is a stochastic process. Using the hedging condition $V(N) = Y$ (which means $V(N, \omega) = Y(\omega)$, for all $\omega \in \Omega_N$)) and (15), we have the following formula for the fair price at time t of the financial derivative:

$$\Pi_Y(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(0), \dots, S(t)]. \tag{18}$$

Equation (18) is known as **risk-neutral pricing formula** and it is the cornerstone of options pricing theory. It holds not only for the binomial model but for any discrete—or even continuum —pricing model for financial derivatives. It is used for standard as well as non-standard European derivatives. In the special case $t = 0$, (18) reduces to

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y]. \tag{19}$$

Remark 0.4. We may interpret (19) as follows: the current (at time $t = 0$) fair value of the derivative is our expectation on the future payment of the derivative (the pay-off) expressed in terms of the future value of money (discounted pay-off $Y^* = e^{-rN}Y$). The expectation has to be taken *with respect to the martingale probability measure*, i.e., ignoring any (subjective or illegal[2]) estimate on future movements of the stock price (except for the loss in value due to the time-devaluation of money).

Example. Consider a 2-period binomial model with the following parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad r = 0, \quad p \in (0,1).$$

Assume further that $S_0 = 36$. Consider the European derivative with pay-off

$$Y = (S(2) - 28)_+ - 2(S(2) - 32)_+ + (S(2) - 36)_+$$

and time of maturity $T = 2$. According to (19), the fair value of the derivative at $t = 0$ is

$$\Pi_Y(0) = e^{-2r}\mathbb{E}_q[Y] = \mathbb{E}_q[(S(2) - 28)_+] - 2\mathbb{E}_q[(S(2) - 32)_+] + \mathbb{E}_q[(S(2) - 36)_+].$$

By the market parameters we find $q = 1/2$. Hence the distribution of $S(2)$ in the risk-neutral probability measure is

$$\mathbb{P}_q(S(2) = s) = \begin{cases} 1/4 & \text{if } s = 16 \text{ of } s = 64 \\ 1/2 & \text{if } s = 32 \\ 0 & \text{otherwise} \end{cases}.$$

It follows that

$$\mathbb{E}_q[(S(2) - 28)_+] = 11, \quad \mathbb{E}_q[(S(2) - 32)_+] = 8, \quad \mathbb{E}_q[(S(2) - 36)_+] = 7,$$

hence $\Pi_Y(0) = 2$.

By definition of expectation in the N-coin toss probability space, see (2), the risk-neutral pricing formula (19) for the standard European derivative with pay-off $Y = g(S(N))$ and maturity $T = N$ takes the explicit form

$$\Pi_Y(0) = e^{-rN} \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^k g(S_0 e^{ku+(N-k)d}).$$

However this formula is not very convenient for numerical computations, because the binomial coefficient $\binom{N}{k}$ will reach very large values for even a relative small number of steps (e.g., $\binom{50}{25}$ is of order 10^{14}). A much more convenient way to compute numerically the binomial price of standard European derivatives is by using the recurrence formula $\Pi_Y(N) = Y$ and

$$\Pi_Y(t) = e^{-r}(q\Pi_Y^u(t+1) + (1-q)\Pi_Y^d(t+1)), \quad t = 0, \dots, N-1, \tag{20}$$

[2]Trading in the market using privileged information is a crime (**insider trading**).

where $\Pi_Y^u(t)$ is the binomial price of the derivative at time t assuming that the stock price goes up at time t, i.e.,

$$\Pi_Y^u(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(0),\dots,S(t-1),S(t)=S(t-1)e^u]$$

and similarly one defines $\Pi_Y^d(t)$ by replacing "up" with "down". The formula (20) follows immediately by (18) and the definition of conditional expectation.

Remark 0.5. It can be shown that any European derivative in the binomial market can be hedged by a self-financing portfolio invested in the underlying stock and the risk-free asset, see [2]. For this reason the binomial market is called a **complete market**. In fact, the **second fundamental theorem of asset pricing** states that market completeness is equivalent to the uniqueness of the risk-neutral probability measure. An arbitrage free market is said to be **incomplete** if the risk-neutral measure is not unique. When the market is incomplete the price of European derivatives is not uniquely defined and moreover there exist European derivatives which cannot be hedged by self-financing portfolios. An example of incomplete market is the trinomial model discussed in the project in Chapter 2.

Implementation of the binomial model

For real world applications the binomial model must be properly rescaled in time. Precisely, let $T>0$ be the maturity of a European derivative and consider the uniform partition of the interval $[0,T]$ with size $h>0$:

$$0=t_0<t_1<\dots<t_N=T, \quad t_i-t_{i-1}=h, \quad \text{for all } i=1,\dots,N.$$

The binomial stock price on the given partition is given by $S(0)=S_0>0$ and

$$S(t_i)=\begin{cases} S(t_{i-1})e^u, & \text{with probability } p,\\ S(t_{i-1})e^d, & \text{with probability } 1-p,\end{cases} \qquad i=1,\dots,N,$$

while

$$B(t_i)=B_0e^{rhi}.$$

The **instantaneous mean of log-return** and the **instantaneous variance** of the binomial stock price are defined respectively by

$$\alpha=\frac{1}{h}\mathbb{E}_p[\log S(t_i)-\log S(t_{i-1})]=\frac{1}{h}[pu+(1-p)d],$$
$$\sigma^2=\frac{1}{h}\mathrm{Var}_p[\log S(t_i)-\log S(t_{i-1})]=\frac{(u-d)^2}{h}p(1-p).$$

The parameter σ itself is called **instantaneous volatility**. Note carefully that these parameters are constant in the standard binomial model and that they are computed with the physical probability (and *not* with the risk-neutral probability). Inverting the equations above we obtain

$$u = \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}. \tag{21}$$

In the applications of the binomial model it is customary to give the parameters α, σ and then compute u, d using (21). The risk-neutral probability then becomes

$$q = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}. \tag{22}$$

The binomial model is trustworthy only for h very small compared to T (i.e., $N >> 1$).

The following Matlab code defines a function `EuroZeroBin(g, T, s, alpha, sigma, r, p, N)` that computes the initial price of the standard European derivative with pay-off $Y = g(S(T))$ using (20). The variable s is the initial price S_0 of the stock. The function also checks that $q \in (0, 1)$, i.e., that the risk-neutral probability is well defined (and thus the market is free of self-financing arbitrages). If not a message appears which asks to increase the number of steps N.

```
function Pzero=EuroZeroBin(g,T,s,alpha,sigma,r,p,N)
h=T/N;
u=alpha*h+sigma*sqrt(h)*sqrt((1-p)/p);
d=alpha*h-sigma*sqrt(h)*sqrt(p/(1-p));
qu=(exp(r*h)-exp(d))/(exp(u)-exp(d));
qd=1-qu;
if (qu<0 || qd<0)
display('Error:  the market is not arbitrage free.  Increase the value of N');
Pzero=0;
return
end
S=zeros(N+1,1);
P=zeros(N+1);
S=s*exp((N-[0:N])*u+[0:N]*d).';
P(:,N+1)=g(S);
for j=N:-1:1
for i=1:j
P(i,j)=exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1));
end
end
```

As shown in [2], the binomial price of the derivative is very weakly dependent on the parameter $\alpha \in \mathbb{R}$ and $p \in (0, 1)$ (provided N is sufficiently large, say $N \approx 10000$). Hence one normally chooses $\alpha = 0$ and $p = 1/2$ in the implementation of the binomial model.

0.3 Probability theory on uncountable sample spaces

In this section we assume that Ω is uncountable (e.g., $\Omega = \mathbb{R}$). In this case there is no general procedure to construct a probability space, but only an abstract definition. In particular a probability measure $\mathbb{P}$ on events $A \subseteq \Omega$ is defined only axiomatically by requiring that $0 \leq \mathbb{P}(A) \leq 1$, $\mathbb{P}(\Omega) = 1$ and that, for any sequence of disjoint events $A_1, A_2, \dots$, it should hold

$$\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$$

Moreover it is not necessary—and almost never convenient—to assume that $\mathbb{P}$ is defined for all events $A \subset \Omega$. We denote by $\mathcal{F}$ the set of events (i.e., subsets of Ω) which have a well defined probability satisfying the properties above.

Example. Let $\Omega = \mathbb{R}$. We say that $A \subseteq \mathbb{R}$ is a **Borel** set if it can be written as the union (or intersection) of countably many open (or closed) intervals. Let $\mathcal{F}$ be the collection of all Borel sets. Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function such that

$$\int_{\mathbb{R}} p(\omega)\, d\omega = 1.$$

Then $\mathbb{P} : \mathcal{F} \to [0, 1]$ given by

$$\mathbb{P}(A) = \int_A p(\omega)\, d\omega \tag{23}$$

defines a probability. If $X : \mathbb{R} \to \mathbb{R}$ is a random variables, the expectation of X in the probability measure (23) is given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} X(\omega) p(\omega)\, dx, \tag{24}$$

provided the integral converges.

Fortunately for most applications (and in particular for those in financial mathematics) the knowledge of the full probability space is usually not necessary, as in the applications one is typically concerned only with random variables and their distributions, rather than with generic events. More precisely, we are only interested in assigning a probability to events of the form $\{X \in I\}$, where X is a random variable on the (abstract) probability space and $I \subset \mathbb{R}$, that is to say, events which can be resolved by one (or more) random variables.

Remark 0.6. Even though Ω is uncountable, the image of $X : \Omega \to \mathbb{R}$ need not be uncountable (e.g., X could be piecewise constant). To avoid technical complications we assume in the following that $\mathrm{Im}(X)$ does not contain isolated points. We shall refer to these random variables as **continuum random variables**. The only case of non-continuum random variable that we allow in this section is when X is a deterministic constant, in which case the image of X consists of one real number only.

The probability $\mathbb{P}(X \in I)$ can be computed explicitly when X has a density.

Definition 0.2. *Let $f_X : \mathbb{R} \to [0, \infty)$ be a continuous function, except possibly on finitely many points. A continuum random variable $X : \Omega \to \mathbb{R}$ is said to have* **probability density** f_X *if*

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx,$$

for all Borel sets $A \subseteq \mathbb{R}$.

Note that the density f_X satisfies

$$\int_{\mathbb{R}} f_X(x)\, dx = 1$$

and the **cumulative distribution** $F_X(x) = \mathbb{P}(X \leq x)$ satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy, \quad \text{for all } x \in \mathbb{R}, \text{ hence } f_X = \frac{dF_X}{dx}.$$

Example. A random variable $X : \Omega \to \mathbb{R}$ is said to be a **normal** random variable with **mean** $m \in \mathbb{R}$ and **variance** $\sigma^2 > 0$ if it admits the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-m|^2}{2\sigma^2}\right). \tag{25}$$

We denote $\mathcal{N}(m, \sigma^2)$ the set of all such random variables. A variable $X \in \mathcal{N}(0, 1)$ is called a **standard normal** random variable. The cumulative distribution of standard normal random variables is denoted by $\Phi(x)$ and is called the **standard normal distribution**, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

The following theorem shows that the probability density, when it exists, provides all the relevant statistical information on a random variable.

Theorem 0.7. *The following holds for all sufficiently regular*[3] *functions $g : \mathbb{R} \to \mathbb{R}$:*

(i) Let $X : \Omega \to \mathbb{R}$ be a random variable with density f_X. Then for all Borel sets $A \subseteq \mathbb{R}$,

$$\mathbb{P}(g(X) \in A) = \int_{x: g(x) \in A} f_X(x)\, dx.$$

(ii) Let $X : \Omega \to \mathbb{R}$ be a random variable with density f_X. Then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(y) f_X(y)\, dy.$$

[3] In particular, for all functions g such that the integrals in the theorem are well-defined.

Moreover the properties 1,2,3 in Theorem 0.1 still hold for continuum random variables.

By (ii) in Theorem 0.7, the expectation and the variance of a continuum random variable X with density f_X are given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\, dx, \quad \mathrm{Var}[X] = \int_{\mathbb{R}} x^2 f_X(x)\, dx - \left(\int_{\mathbb{R}} x f_X(x)\, dx\right)^2. \tag{26}$$

Applying (26) to normal variables we obtain

$$X \in \mathcal{N}(m, \sigma^2) \implies \mathbb{E}[X] = m, \quad \mathrm{Var}[X] = \sigma^2. \tag{27}$$

Joint probability density

Definition 0.3. *Two continuum random variables $X, Y : \Omega \to \mathbb{R}$ are said to have the* **joint probability density** $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$, *if*

$$\mathbb{P}(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x, y)\, dx\, dy,$$

for all Borel sets $A, B \subseteq \mathbb{R}$.

Note that if $f_{X,Y}$ is a joint probability density, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_{X,Y}(x, y)\, dx\, dy = 1.$$

Moreover if we define the **joint cumulative distribution** as $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ then

$$f_{X,Y}(x, y) = \partial_x \partial_y F_{X,Y}(x, y).$$

When X, Y have the joint density $f_{X,Y}(x, y)$, the random variables X, Y admit the densities

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dy, \quad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dx.$$

Example: Jointly normally distributed random variables. Let $m \in \mathbb{R}^2$ and $C = (C_{ij})_{i,j=1,2}$ be a symmetric, positive definite 2×2 matrix. Two random variables $X_1, X_2 : \Omega \to \mathbb{R}$ are said to be jointly normally distributed with mean m and **covariance matrix** C if they admit the joint density

$$f_{X_1,X_2}(x) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left(-\frac{1}{2}(x - m)C^{-1}(x - m)\right), \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{28}$$

The following theorem generalizes Theorem 0.7 in the presence of two variables.

Theorem 0.8. *Let $X, Y : \Omega \to \mathbb{R}$ be random variables with joint density f_X and $g : \mathbb{R}^2 \to \mathbb{R}$.*

(i) For all Borel sets $A \subseteq \mathbb{R}$ there holds

$$\mathbb{P}(g(X,Y) \in A) = \int_{(x,y):g(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy.$$

(ii) There holds

$$\mathbb{E}[g(X,Y)] = \int_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y)\,dx\,dy.$$

By (ii) of Theorem 0.8, if X_1, X_2 have the joint density f_{X_1,X_2}, then the covariance of X_1, X_2 can be computed as

$$\begin{aligned}\mathrm{Cov}(X_1,X_2) &= \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\ &= \int_{\mathbb{R}^2} x_1x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 \\ &\quad - \int_{\mathbb{R}^2} x_1 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 \int_{\mathbb{R}^2} x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2.\end{aligned}$$

In particular, if X_1, X_2 are jointly normal distributed with mean $m \in \mathbb{R}^2$ and covariance matrix $C = (C_{ij})_{i,j=1,2}$, we find

$$m = (m_1, m_2), \qquad C_{ij} = \mathrm{Cov}(X_i, X_j). \tag{29}$$

The following result on the linear combination of independent normal random variables will play an important role for the project in multi-asset options in Chapter 5.

Theorem 0.9. *Let $X_1, X_2 \in \mathcal{N}(0,1)$ be independent and $a, b, c, d \in \mathbb{R}$. Then $aX_1 + bX_2 \in \mathcal{N}(0, a^2 + b^2)$. Moreover if*

$$Y_1 = aX_1 + bX_2, \quad Y_2 = cX_1 + dX_2,$$

and if the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible, then Y_1, Y_2 are jointly normally distributed with zero mean and covariant matrix $C = AA^T$.

Stochastic processes. Martingales

Let Ω be an uncountable sample space. A stochastic process is a one parameter family $\{X(t)\}_{t\geq 0}$ of (continuum) random variables $X(t) : \Omega \to \mathbb{R}$. We denote $X(t,\omega) = X(t)(\omega)$.

The parameter t is referred to as the time variable, since this is what it represents in the applications that we have in mind. For each $\omega \in \Omega$ *fixed*, the function $t \to X(t, \omega)$ is called a path of the stochastic process. If the paths are all the same for all $\omega \in \Omega$, then we say that $X(t)$ is a deterministic function of time.

Martingale stochastic processes play a fundamental role in options pricing theory[4]. To define martingales on uncountable sample spaces, let $\mathcal{F}_X(t)$ denote the information accumulated by "looking" at the stochastic process up to time t, i.e., the collection of events resolved by $X(s)$ for $0 \le s \le t$. Intuitively, the stochastic process $\{X(t)\}_{t\ge 0}$ is a martingale if, based on the information contained in $\mathcal{F}_X(s)$, our "best estimate" on $X(t)$ for $t > s$ is $X(s)$, i.e., we are not able to estimate whether the process will raise or fall in the interval $[s, t]$ with the information available at time s. This intuitive definition is encoded in the formula

$$\mathbb{E}[X(t)|\mathcal{F}_X(s)] = X(s), \quad 0 \le s \le t, \tag{30}$$

which generalizes the definition (3) of martingales in finite probability theory. The left hand side of (30) is the conditional expectation of $X(t)$ with respect to the information $\mathcal{F}_X(s)$, whose precise definition is not needed here. It can be shown that (30) implies that martingales have constant expectation.

Brownian motion

Next we recall the definition of the most important of all stochastic processes.

Definition 0.4. *A* **Brownian motion**, *or* **Wiener process**, *is a stochastic process* $\{W(t)\}_{t\ge 0}$ *with the following properties:*

1. *For all*[5] $\omega \in \Omega$, *the paths are continuous (i.e.,* $t \to W(t, \omega)$ *is a continuous function) and* $W(0, \omega) = 0$;

2. *For all* $0 = t_0 < t_1 < t_2 < \dots$, *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1), \dots,$$

   are independent random variables and

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0, \quad \mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i, \quad \textit{for all } i = 0, 1, \dots;$$

3. *The increments are normally distributed, that is to say, for all* $0 \le s < t$,

$$\mathbb{P}(W(t) - W(s) \in A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{y^2}{2(t-s)}}\, dy,$$

   for all Borel sets $A \subseteq \mathbb{R}$.

[4]In fact, this theory is also called **martingale pricing theory** in some literature.

[5]More precisely, for all $\omega \in \Omega$ up to a set of zero probability.

It can be shown that Brownian motions exist, yet a formal construction is technically quite difficult and beyond the purpose of this text.

Remark 0.7. Since the definition of Brownian motion depends on the probability measure $\mathbb{P}$, then a stochastic process $\{W(t)\}_{t\geq 0}$ which is a Brownian motion in the probability measure $\mathbb{P}$ will in general *not* be a Brownian motion in another probability measure $\widetilde{\mathbb{P}}$. When we want to emphasize that $\{W(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}$, we shall say that $\{W(t)\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.

Remark 0.8. Letting $s = 0$ in property 3 in Definition 0.4 we obtain that $W(t) \in \mathcal{N}(0, t)$, for all $t > 0$. In particular, $W(t)$ has zero expectation for all times. It can also be shown that Brownian motions are martingales.

The following result is used a few times in the following chapters.

Theorem 0.10. *Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let*

$$X(t) = g(t)W(t) - \int_0^t g'(s)W(s)\, ds.$$

Then

$$X(t) \in \mathcal{N}(0, \Delta(t)), \quad \Delta(t) = \int_0^t g(s)^2\, ds.$$

Sketch of the proof. We have

$$\mathbb{E}[X(t)] = g(t)\mathbb{E}[W(t)] - \int_0^t g'(s)\mathbb{E}[W(s)]\, ds = 0,$$

$$\begin{aligned}
\mathrm{Var}[X(t)] = \mathbb{E}[X(t)^2] &= g(t)^2\mathbb{E}[W(t)^2] + \mathbb{E}\left[\left(\int_0^t g'(s)W(s)\, ds\right)^2\right] \\
&- 2g(t)\mathbb{E}\left[\int_0^t g'(s)W(t)W(s)\, ds\right] \\
&= g(t)^2 t + \int_0^t \int_0^t g'(s)g'(\tau)\mathrm{Cov}(W(s), W(\tau))\, d\tau\, ds \\
&- 2g(t)\int_0^t g'(s)\mathrm{Cov}(W(s), W(t))\, ds.
\end{aligned}$$

Using $\mathrm{Cov}(W(s), W(t)) = \min(s, t)$, and after some technical but straightforward calculation, we obtain $\mathrm{Var}[X(t)] = \Delta(t)$. To show that $X(t)$ is normally distributed, let $\{t_0 = 0, \dots, t_n = t\}$ be a uniform partition of the interval $[0, t]$ and consider the Riemann sum approximation of $X(t)$:

$$X_n(t) = g(t_n)W(t_n) - \sum_{i=1}^{n}(g(t_i) - g(t_{i-1})W(t_i)$$
$$= -\sum_{i=1}^{n-1} g(t_i)W(t_i) + \sum_{i=1}^{n} g(t_{i-1})W(t_i)$$
$$= -\sum_{i=0}^{n-1} g(t_i)W(t_i) + \sum_{j=0}^{n-1} g(t_j)W(t_{j+1}),$$

where in the last step we used $W(t_0) = W(0) = 0$ in the first sum and made the change of index $j = i - 1$ in the second sum. Hence

$$X_n(t) = \sum_{i=0}^{n-1} g(t_i)(W(t_{i+1}) - W(t_i)).$$

Thus $X_n(t)$ is normally distributed because it is a linear combination of the independent and normally distributed random variables $W(t_{i+1}) - W(t_i)$. It can be shown that this property carries over in the limit $n \to \infty$ and since $X_n(t) \to X(t)$ in this limit the proof is completed. □

Remark 0.9. By using the formal identity $d(g(t)W(t)) = g'(t)W(t)dt + g(t)dW(t)$, as well as $\int_0^t d(g(s)W(s)) = g(t)W(t)$, we can write the definition of $X(t)$ in Theorem 0.10 as

$$X(t) = \int_0^t g(s)dW(s),$$

which is called **Itô integral** of the deterministic function $g(t)$.

Equivalent probability measures. Girsanov theorem

One further technical complication arising for uncountable sample spaces is the existence of non-trivial events with zero measure, e.g., the event $\{W(t) = 0\}$ that the Brownian motion $W(t)$ takes value zero when $t > 0$. We shall need to consider the concept of equivalent probability measures:

Definition 0.5. *Two probability measure $\mathbb{P}$, $\widetilde{\mathbb{P}}$ on the events $A \in \mathcal{F}$ are said to be equivalent if $\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$.*

Hence equivalent probability measures agree on which events are impossible. Note that in a finite probability space all probability measures are equivalent, as in the finite case the empty set is the only event with zero probability. The following important theorem characterizes the relation between equivalent probability measures on uncountable sample spaces and is known as the Radon-Nikodým theorem. We denote $\mathbb{I}_A$ the **characteristic function** of the set $A \in \mathcal{F}$, i.e., the random variable taking value $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and zero otherwise.

Theorem 0.11 (Radon-Nikodým theorem)**.** *Let $\mathbb{P} : \mathcal{F} \to [0,1]$ be a probability measure. Then $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ is a probability measure equivalent to $\mathbb{P}$ if and only if there exists a random variable $Z : \Omega \to \mathbb{R}$ such that $Z > 0$ (with probability 1), $\mathbb{E}[Z] = 1$ and $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$. Moreover if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent then $\widetilde{\mathbb{E}}[X] = \mathbb{E}[ZX]$, for all random variables $X : \Omega \to \mathbb{R}$.*

For example, assume $\Omega = \mathbb{R}$ and that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are defined as in (23), namely

$$\mathbb{P}(A) = \int_A p(\omega)\, d\omega, \quad \widetilde{\mathbb{P}}(A) = \int_A \widetilde{p}(\omega)\, d\omega,$$

where A is a Borel set and p, $\widetilde{p}$ are two continuous non-negative functions such that

$$\int_{\mathbb{R}} p(\omega)\, d\omega = \int_{\mathbb{R}} \widetilde{p}(\omega)\, d\omega = 1.$$

Then, according to Theorem 0.11 and (24), $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if and only if there exists a function $Z : \mathbb{R} \to \mathbb{R}$ such that $Z > 0$, and

$$\widetilde{\mathbb{P}}(A) = \int_A \widetilde{p}(\omega)\, d\omega = \int_{\mathbb{R}} Z(\omega)\mathbb{I}_A(\omega)p(\omega)\, d\omega = \int_A Z(\omega)p(\omega)\, d\omega.$$

As the equality $\int_A \widetilde{p}(\omega)\, d\omega = \int_A Z(\omega)p(\omega)\, d\omega$ has to be satisfied for all Borel sets $A \subset \mathbb{R}$, then $\widetilde{p}(\omega) = Z(\omega)p(\omega)$ must hold for all $\omega \in \mathbb{R}$ (up to a set with zero probability).

Theorem 0.12 (and Definition). *Let $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion. Given $\theta \in \mathbb{R}$ and $T > 0$ define*

$$Z_\theta = e^{-\theta W(T) - \frac{1}{2}\theta^2 T}. \tag{31}$$

Then $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ defines a probability measure equivalent to $\mathbb{P}$, which is called **Girsanov's probability** *with parameter $\theta \in \mathbb{R}$. for all Borel sets $A \subseteq \mathbb{R}$.*

Proof. The proof follows immediately from Theorem 0.11, since the random variable (31) satisfies $Z_\theta > 0$ and

$$\mathbb{E}[Z_\theta] = \mathbb{E}[e^{-\theta W(T) - \frac{1}{2}\theta^2 T}] = \int_{\mathbb{R}} e^{-\theta x - \frac{1}{2}\theta^2 T} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}\, dx = 1,$$

where we used the density of the normal random variable $W(T) \in \mathcal{N}(0,T)$ to compute the expectation of Z_θ in the probability measure $\mathbb{P}$ (see Theorem 0.7(ii)). □

Note that the Girsanov probability measure $\mathbb{P}_\theta$ depend also on T, but this is not reflected in our notation. In the following we denote by $\mathbb{E}_\theta[\cdot]$ the expectation computed in the probability measure $\mathbb{P}_\theta$ for $\theta \neq 0$. When $\theta = 0$ then $\mathbb{P}_\theta = \mathbb{P}$, in which case the expectation is denoted as usual by $\mathbb{E}[\cdot]$. By Theorem 0.11 we have $\mathbb{E}_\theta[X] = \mathbb{E}[Z_\theta X]$, for all random variables

$X : \Omega \to \mathbb{R}$. Moreover we now show that $\mathbb{E}_\theta[W(t)] = -\theta t$. In fact by the Radon-Nikodým theorem we have

$$\mathbb{E}_\theta[W(t)] = \mathbb{E}[Z_\theta W(t)] = \mathbb{E}[e^{-\theta W(T) - \frac{1}{2}\theta^2 T} W(t)].$$

Adding and subtracting $W(t)$ in the exponent of the exponential function we have

$$\mathbb{E}_\theta[W(t)] = \mathbb{E}[e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T} e^{-\theta W(t)} W(t)] = \mathbb{E}[e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T}]\mathbb{E}[e^{-\theta W(t)} W(t)],$$

where in the last step we used that the random variables $X = e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T}$ and $Y = e^{-\theta W(t)} W(t)$ are independent (being functions of the independent random variables $W(T) - W(t)$ and $W(t)$). Using $W(T) - W(t) \in \mathcal{N}(0, T-t)$ and $W(t) \in \mathcal{N}(0, t)$, we can compute the expectations of X and Y as

$$\mathbb{E}[X] = e^{-\frac{1}{2}\theta^2 T} \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} e^{-\theta x - \frac{x^2}{2(T-t)}}\, dx = e^{-\frac{\theta^2}{2} t},$$

$$\mathbb{E}[Y] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\theta x - \frac{x^2}{2t}} x\, dx = -e^{\frac{\theta^2}{2} t} \theta t.$$

Hence $\mathbb{E}_\theta[W(t)] = \mathbb{E}[X]\mathbb{E}[Y] = -\theta t$, as claimed. It follows that $\{W(t)\}_{t \geq 0}$ is *not* a $\mathbb{P}_\theta$-Brownian motion, since Brownian motions, by definition, have zero expectation at any time. Now we can state a fundamental theorem in probability theory with deep applications in financial mathematics, namely Girsanov's theorem[6].

Theorem 0.13. *Let $\{W(t)\}_{t \geq 0}$ be a $\mathbb{P}$-Brownian motion. Given $\theta \in \mathbb{R}$ and $T > 0$, let $\mathbb{P}_\theta$ be the Girsanov probability measure with parameter θ introduced in Theorem 0.12. Define the stochastic process $\{W^{(\theta)}(t)\}_{t \geq 0}$ by*

$$W^{(\theta)}(t) = W(t) + \theta t. \tag{32}$$

Then $\{W^{(\theta)}(t)\}_{t \geq 0}$ is a $\mathbb{P}_\theta$-Brownian motion.

Note carefully that $\{W^{(\theta)}(t)\}_{t \geq 0}$ is *not* a $\mathbb{P}$-Brownian motion, as it follows by the fact that $\mathbb{E}[W^{(\theta)}(t)] = \theta t$. In particular, according to the probability measure $\mathbb{P}$, the stochastic process $\{W^{(\theta)}(t)\}_{t \geq 0}$ has a *drift*, i.e., a tendency to move up (if $\theta > 0$) or down (if $\theta < 0$). However in the Girsanov probability this drift is removed, because, as shown before, $\mathbb{E}_\theta[W^{(\theta)}(t)] = \mathbb{E}_\theta[W(t)] + \theta t = 0$.

Multi-dimensional Girsanov theorem

We conclude this section with a generalization of Girsanov's theorem in the presence of two independent Brownian motions. This generalization is important for the project on multi-asset options in Chapter 5. We limit ourselves to state without proof the analogs of Theorems 0.12 and 0.13 required for this purpose.

[6] Actually we consider only a special case of this theorem, which suffices for our purposes.

Theorem 0.14 (and Definition). *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be $\mathbb{P}$-independent Brownian motions. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $T > 0$ define*

$$Z_\theta = e^{-\theta_1 W_1(T) - \theta_2 W_2(T) - \frac{1}{2}(\theta_1^2 + \theta_2^2)T}. \tag{33}$$

Then $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ defines a probability measure equivalent to $\mathbb{P}$, which is called Girsanov's probability with parameters $\theta_1, \theta_2 \in \mathbb{R}$.

Theorem 0.15. *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be $\mathbb{P}$-independent Brownian motions. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $T > 0$, let $\mathbb{P}_\theta$ be the Girsanov probability with parameters θ_1, θ_2 introduced in Theorem 0.14. Define the stochastic processes $\{W_1^{(\theta)}(t)\}_{t\geq 0}$, $\{W_2^{(\theta)}(t)\}_{t\geq 0}$ by*

$$W_1^{(\theta)}(t) = W_1(t) + \theta_1 t, \quad W_2^{(\theta)}(t) = W_2(t) + \theta_2 t \tag{34}$$

Then $\{W_1^{(\theta)}(t)\}_{t\geq 0}$, $\{W_2^{(\theta)}(t)\}_{t\geq 0}$ are $\mathbb{P}_\theta$-independent Brownian motions.

0.4 Black-Scholes options pricing theory

In the binomial model the stock price at time t is a finite random variable $S(t)$. In the Black-Scholes model the stock price is a continuum random variable with image $\mathrm{Im}(S(t)) = (0, \infty)$, namely the **geometric Brownian motion**

$$S(t) = S_0 e^{\alpha t + \sigma W(t)}. \tag{35}$$

The probability $\mathbb{P}$ with respect to which $\{W(t)\}_{t\geq 0}$ is Brownian motion is the **physical** (or **real-world**) **probability** of the Black-Scholes market. Moreover α is the **instantaneous mean of log-return**, σ is the **instantaneous volatility** and σ^2 is the **instantaneous variance** of the geometric Brownian motion

The geometric Brownian motion admits the density

$$f_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right), \tag{36}$$

where $H(x)$ is the **Heaviside** function. It can be shown that the binomial stock price converges in distribution to the geometric Brownian motion in the time-continuum limit, see [2].

The risk-neutral pricing formula in Black-Scholes markets

The purpose of this section is to introduce the definition of Black-Scholes price of European derivatives from a probability theory point of view. Recall that the probabilistic formulation of the binomial options pricing model is encoded in the risk-neutral pricing formula (18). Our goal is to derive a similar risk-neutral pricing formula (at time $t = 0$) for the time-continuum Black-Scholes model.

Motivated by the approach for the binomial model, we first look for a probability measure in which the the discounted stock price in Black-Scholes markets is a martingale (martingale probability measure). It is natural to seek such martingale probability within the class of Girsanov probabilities $\mathbb{P}_\theta$ equivalent to the physical probability $\mathbb{P}$ which we defined in Theorem 0.12. To this purpose we shall need the form of the density function of the geometric Brownian motion in the probability measure $\mathbb{P}_\theta$.

Theorem 0.16. *Let $\theta \in \mathbb{R}$, $T > 0$ and $\mathbb{P}_\theta$ be the Girsanov probability measure equivalent to the physical probability $\mathbb{P}$. The geometric Brownian motion* (35) *has the following density in the probability measure $\mathbb{P}_\theta$:*

$$f^{(\theta)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right). \tag{37}$$

Proof. Since

$$S(t) = S_0 e^{\alpha t + \sigma W(t)} = S_0 e^{(\alpha-\theta\sigma)t + \sigma W^{(\theta)}(t)}, \quad W^{(\theta)}(t) = W(t) + \theta t$$

and since $\{W^{(\theta)}(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}_\theta$ (see Girsanov's Theorem 0.13), then the density $f^{(\theta)}_{S(t)}$ is the same as $f_{S(t)}$ with α replaced by $\alpha - \theta\sigma$. □

Let $\mathbb{E}_\theta[\cdot]$ denote the expectation in the measure $\mathbb{P}_\theta$. Recall that martingales have constant expectation. Hence in the martingale (or risk-neutral) probability measure the expectation of the discounted value of the stock must be constant, i.e., $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$. This condition alone suffices to single out a unique possible value of θ.

Theorem 0.17. *The identity $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ holds if and only if $\theta = q$, where*

$$q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}. \tag{38}$$

Proof. Using the density (37) of $S(t)$ in the measure $\mathbb{P}_\theta$ and (26) we have

$$\mathbb{E}_\theta[S(t)] = \int_{\mathbb{R}} x f^{(\theta)}_{S(t)}(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2 t}}\int_0^\infty \exp\left(-\frac{(\log x - \log S_0 - (\alpha-\theta\sigma)t)^2}{2\sigma^2 t}\right) dx.$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha-\theta\sigma)t}{\sigma\sqrt{t}}$, $dx = x\sigma\sqrt{t}\,dy$, we obtain

$$\mathbb{E}_\theta[S(t)] = \frac{S_0}{\sqrt{2\pi}} e^{(\alpha-\theta\sigma)t}\int_{\mathbb{R}} e^{-\frac{y^2}{2}+\sigma\sqrt{t}y}\,dy = S_0 e^{(\alpha-\theta\sigma+\frac{\sigma^2}{2})t}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{(y+\sigma\sqrt{t})^2}{2}}\,dy.$$

As $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = 1$, the result follows. □

Even though the validity of $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ is only necessary for the discounted geometric Brownian motion to be a martingale, one can show that the following result holds.

Theorem 0.18. *The discounted value of the geometric Brownian motion stock price is a martingale in the probability measure $\mathbb{P}_\theta$ if and only if $\theta = q$, where q is given by* (38).

The previous discussion leads us to the following definition.

Definition 0.6. *Given $\alpha \in \mathbb{R}$, $\sigma > 0$, $r \in \mathbb{R}$ and $T > 0$, the probability measure*

$$\mathbb{P}_q(A) = \mathbb{E}[e^{-\theta W(T) - \frac{1}{2}\theta^2 T}\mathbb{I}_A], \quad q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}$$

is called the **martingale probability**, *or* **risk-neutral probability**, *in the interval* $[0, T]$ *of the Black-Scholes market with parameters α, σ, r.*

Remark 0.10. In the risk-neutral probability the stock price is given by the geometric Brownian motion

$$S(t) = S(0)e^{(r - \frac{\sigma^2}{2})t + \sigma W^{(q)}(t)}, \tag{39}$$

where, by Girsanov's theorem, $W^{(q)}(t) = W(t) + qt$ is a Brownian motion in the risk-neutral probability. This follows by replacing $\alpha = r + q\sigma - \frac{1}{2}\sigma^2$ into (35).

At this point we have all we need to define the Black-Scholes price of European derivatives at time $t = 0$ using the risk-neutral pricing formula.

Definition 0.7. *The Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula*

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y], \tag{40}$$

i.e., it equals the expected value of the discounted pay-off in the risk-neutral probability measure of the Black-Scholes market.

In the case of standard European derivatives we can use the density of the geometric Brownian motion in the risk-neutral probability measure to write the Black-Scholes price in the following integral form.

Theorem 0.19. *For the standard European derivative with pay-off $Y = g(S(T))$ at maturity $T > 0$, the Black-Scholes price at time $t = 0$ can be written as $\Pi_Y(0) = v_0(S_0)$, where S_0 is the price of the underlying stock at time $t = 0$ and $v_0 : (0, \infty) \to \mathbb{R}$ is the* **pricing function** *of the derivative at time $t = 0$, which is given by*

$$v_0(x) = e^{-rT}\int_{\mathbb{R}} g(xe^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}y})e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}. \tag{41}$$

Proof. Replacing $\theta = q$ in (37) we obtain that the geometric Brownian motion has the following density in the risk-neutral probability measure $\mathbb{P}_q$:

$$f^{(q)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right). \tag{42}$$

Using the density (42) for $t = T$ in the risk-neutral pricing formula (40) we obtain

$$\begin{aligned}\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y] = e^{-rT}\mathbb{E}_q[g(S(T))] &= \int_{\mathbb{R}} g(x) f^{(q)}_{S(T)}(x)\,dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 t}}\int_0^\infty \frac{g(x)}{x}\exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right) dx.\end{aligned}$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}$ we obtain

$$\Pi_Y(0) = e^{-rT}\int_{\mathbb{R}} g(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}y}) e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} = v_0(S_0),$$

as claimed. □

Remark 0.11. Of course we are tacitly assuming that the pay-off function g is such that the integral in the right hand side of (41) is finite.

For instance, in the case of the European call option with strike K and maturity T, for which the pay-off function is $g(z) = (z - K)_+$, Theorem 0.19 gives

$$\Pi_{\text{call}}(0) = C_0(S_0, K, T), \quad C_0(x, K, T) = x\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{43a}$$

where Φ is the standard normal distribution and

$$d_2 = \frac{\log\frac{x}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_1 = d_2 + \sigma\sqrt{T}. \tag{43b}$$

Definition 0.7 is only valid at time $t = 0$. The risk-neutral pricing formula for $t > 0$ is

$$\Pi_Y(t) = e^{-r(T-t)}\mathbb{E}_q[Y|\mathcal{F}_S(t)], \tag{44}$$

which generalizes (18) to the time continuum case. The right hand side of (44) is the expectation of the discounted pay-off in the risk-neutral probability measure conditional to the information available at time t, which in a Black-Scholes market is determined by the history of the stock price up to time t. It can be shown that in the case of the standard European derivative with pay-off $Y = g(S(T))$ at maturity T, the risk-neutral pricing formula (44) entails that the Black-Scholes price at time $t \in [0, T]$ can be written in the integral form

$$\Pi_Y(t) = v(t, S(t)), \quad \text{where} \quad v(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau}e^{\sigma\sqrt{\tau}\,y}\right)e^{-\frac{y^2}{2}}\,dy, \quad \tau = T - t. \tag{45}$$

Hence the pricing function $v(t, x)$ of the derivative at time t is the same as the pricing function (41) at time $t = 0$ but with maturity T replaced by the time τ left to maturity, which is rather intuitive.

0.5 The Monte Carlo method

The Monte Carlo method is, in its simplest form, a numerical method to compute the expectation of a random variable. Its mathematical validation is based on the **Law of Large Numbers**, which states the following: Suppose $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with expectation $\mathbb{E}[X_i] = \mu$. Then the sample average of the first n components of the sequence, i.e.,

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n),$$

converges (in probability) to μ as $n \to \infty$.

The law of large numbers can be used to justify the fact that if we are given a large number of independent trials $X_1, \ldots, X_n$ of the random variable X, then

$$\mathbb{E}[X] \approx \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

To measure how reliable is the approximation of $\mathbb{E}[X]$ given by the sample average, consider the standard deviation of the trials $X_1, \ldots, X_n$:

$$s_X = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\overline{X} - X_i)^2}.$$

A simple application of the Central Limit Theorem proves that the random variable

$$\frac{\mu - \overline{X}}{s_X/\sqrt{n}}$$

converges in distribution to a standard normal random variable. We use this result to show that the true value μ of $\mathbb{E}[X]$ has about 95% probability to be in the interval

$$[\overline{X} - 1.96\frac{s}{\sqrt{n}}, \overline{X} + 1.96\frac{s}{\sqrt{n}}].$$

Indeed, for n large,

$$\mathbb{P}\left(-1.96 \leq \frac{\mu - \overline{X}}{s_X/\sqrt{n}} \leq 1.96\right) \approx \int_{-1.96}^{1.96} e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}} \approx 0.95.$$

An application to Black-Scholes theory

Using the Monte Carlo method and the risk-neutral pricing formula (19), we can approximate the Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y and maturity $T > 0$ with the sample average

$$\Pi_Y(0) = e^{-rT}\frac{Y_1 + \ldots Y_n}{n}, \tag{46}$$

where $Y_1, \dots, Y_n$ is a large number of independent trials of the pay-off. Each trial Y_i is determined by a path of the stock price. Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$, we may construct a sample of n paths of the geometric Brownian motion on the given partition with the following simple Matlab function:

```
function Path=StockPath(s,sigma,r,T,N,n)
h=T/N;
W=randn(n,N);
q=ones(n,N);
Path=s*exp((r-sigma^2/2)*h.*cumsum(q')+sigma*sqrt(h)*cumsum(W'));
Path=[s*ones(1,n);Path];
```

Note carefully that the stock price is modeled as a geometric Brownian motion with mean of log return $\alpha = r - \sigma^2/2$, which means that the geometric Brownian motion is risk-neutral, see (42). This is of course correct, since the expectation in (46) that we want to compute is in the risk-neutral probability measure. The following Matlab code compute the Black-Scholes price of a call option using the Monte Carlo method. The code also computes the statistical error

$$\text{Err} = 1.96 \frac{s}{\sqrt{n}} \tag{47}$$

of the Monte Carlo price, where s is the standard deviation of the pay-off trials.

```
function [price, conf95]=MonteCarloCall(s,sigma,r,K,T,N,n)
tic
stockPath=StockPath(s,sigma,r,T,N,n);
payOff=max(0,stockPath(N,:)-K);
price=exp(-r*T)*mean(payOff);
conf95=1.96*std(payOff)/sqrt(n);
toc
```

For instance, by running the command

$$[\texttt{price}, \texttt{conf95}] = \texttt{MonteCarloCall}(10, 0.5, 0.01, 10, 1, 100, 100000)$$

we obtain the output

```
price = 1.9976
conf95 = 0.0249
```

The calculation took about half a second. The exact price for the given call obtained by using the formula (43) is 2.0144, which lies within the confidence interval $[1.9976 - 0.0249, 1.9976 + 0.0249] = [1.9727, 2.0225]$ of the Monte Carlo price. Remark: The formula (43) is implemented in Matlab by the function `blsprice`.

Control variate Monte Carlo

The Monte Carlo method just described is also known as **crude** Monte Carlo and can be improved in a number of ways. For instance, it follows by (47) that in order to reduce the error of the Monte Carlo price, one needs to either (i) increase the number of trials n or (ii) reduce the standard derivation s. As increasing n can be very costly in terms of computational time, the approach (ii) is preferable. There exist several methods to decrease the standard deviation of a Monte Carlo computation, which are collectively called **variance reduction techniques**. Here we describe the **control variate** method.

Suppose we want to compute $\mathbb{E}[X]$. The idea of the control variate method is to introduce a second random variable Q for which $\mathbb{E}[Q]$ can be computed *exactly* and then write

$$\mathbb{E}[X] = \mathbb{E}[Y] + \mathbb{E}[Q], \quad \text{where } Y = X - Q.$$

Hence the Monte Carlo approximation of $\mathbb{E}[X]$ can now be written as

$$\mathbb{E}[X] \approx \frac{Y_1 + \cdots + Y_n}{n} + \mathbb{E}[Q],$$

where $Y_1, \ldots, Y_n$ are independent trials of the random variable Y. This approximation improves the crude Monte Carlo estimate (without control variate) if the sample average estimator of $\mathbb{E}[Y]$ is better than the sample average estimator of $\mathbb{E}[X]$. Because of (47), this will be the case if $(s_Y)^2 < (s_X)^2$. It will now be shown that the latter inequality holds if *X, Q have a positive large correlation.* Letting $X_1, \ldots, X_n$ be independent trials of X and $Q_1, \ldots, Q_n$ be independent trials of Q, we compute

$$\begin{aligned}(s_Y)^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(\overline{Y} - Y_i)^2 = \frac{1}{n-1}\sum_{i=1}^{n}((\overline{X} - \overline{Q}) - (X_i - Q_i))^2 \\ &= (s_X)^2 + (s_Q)^2 - 2C(X, Q),\end{aligned}$$

where $C(X, Q)$ is the sample covariance of the trials $(X_1, \ldots, X_n)$, $(Q_1, \ldots, Q_n)$, namely

$$C(X, Q) = \sum_{i=1}^{n}(\overline{X} - X_i)(\overline{Q} - Q_i).$$

Hence $(s_Y)^2 < (s_X)^2$ holds provided $C(X, Q)$ is sufficiently large and positive (precisely, $C(X, Q) > s_Q/\sqrt{2}$). As $C(X, Q)$ is an unbiased estimator of $\mathrm{Cov}(X, Q)$, then the use of the control variate Q will improve the performance of the crude Monte Carlo method if X, Q have a positive large correlation. An application of this method to the Asian option is one of the goals of the project in Chapter 3.

Chapter 1

A project on the binomial model with stochastic interest rate

In Section 0.2 we have discussed the binomial options pricing model under the assumption that the risk-free asset has constant interest rate. In this section we consider a binomial market in which the interest rate of the risk-free asset is a stochastic process. For the sake of concreteness we model the interest rate by the Ho-Lee model. This working example should be enough to grasp the general theory.

1.1 The generalized binomial model

Let $\{M_t\}_{t=0,\dots,N}$ be a N-period *generalized* random walk with transition probabilities

$$\mathbb{P}(m_{t-1} \to m_t) = \begin{cases} p_t(m_{t-1}) & \text{if } m_t = m_{t-1} + 1 \\ 1 - p_t(m_{t-1}) & \text{if } m_t = m_{t-1} - 1 \\ 0 & \text{otherwise} \end{cases} . \tag{1.1}$$

We consider a binomial market consisting of a stock with price

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t\right], \quad t = 0, 1, 2, \dots N, \tag{1.2a}$$

together with a risk-free asset with value

$$B(t) = B(t-1)(1 + R(t-1)), \quad t = 1, 2, \dots N,$$

where $\{R(t)\}_{t=0,\dots,N-1}$, $R(t) > -1$, is the discretely compounded interest rate process. By iterating the previous equation it follows that

$$B(t) = B_0 \prod_{k=0}^{t-1}(1 + R(k)), \tag{1.2b}$$

where $B_0 > 0$ is the initial value of the risk-free asset. As $1 + R(t) > 0$, then $B(t) > 0$ for all $t = 0, \dots, N$.

Remark 1.1. We may also introduce the continuously compounded risk-free rate process $\{r(t)\}_{t=0,\dots,N-1}$ through the formula $r(t) = \log(1 + R(t))$, $t = 0, \dots, N-1$. In terms of $r(t)$, we can write (1.2b) as $B(t) = B_0 \exp \sum_{k=0}^{t-1} r(k)$, which in the case $r(t) = r$ =constant reduces to the formula $B(t) = B_0 e^{rt}$ used in Section 0.2. For the study of the stochastic risk-free rate binomial model it is preferable to work with the process $\{R(t)\}_{t=0,\dots,N-1}$.

In the following we assume that the risk-free process $\{R(t)\}_{t=0,\dots,N-1}$ is measurable with respect to the generalized random walk $\{M_t\}_{t=0,\dots,N}$. In particular the stochastic process $\{M_t\}_{t=0,\dots,N}$ completely defines the state of the market.

The discounted value (at time $t = 0$) of the stock in the market (1.2) is defined as $S^*(t) = \frac{B_0}{B(t)} S(t)$, that is

$$S^*(t) = D(t)S(t) = \frac{S(t)}{(1 + R(0))(1 + R(1)) \dots (1 + R(t-1))}, \tag{1.3}$$

where

$$D(0) = 1, \quad D(t) = \prod_{k=0}^{t-1} (1 + R(k))^{-1}, \quad t = 1, \dots, N \tag{1.4}$$

is the **discount process**. The market (1.2) is arbitrage free if there exist transition probabilities (1.1) which make the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ a martingale; if this martingale probability is unique, the market is complete. We discuss below one example.

The Ho-Lee model

The literature abounds of stochastic models for the risk-free rate, see [1]. In this chapter we shall study the (discrete) **Ho-Lee model**:

$$R(t) = a(t) + b(t)M_t, \qquad \text{where } a(t) \in \mathbb{R} \text{ and } b(t) > 0, \quad t = 0, 1, 2, \dots, N-1. \tag{1.5}$$

Since the minimum value of M_t is $-t$, then the condition $R(t) > -1$ is satisfied along all paths if and only if

$$a(t) > b(t)t - 1, \tag{1.6}$$

which will be assumed from now on. Our purpose is to prove that the market (1.2), with the risk-free rate given by the Ho-Lee model, is complete under simple conditions on the market parameters.

Theorem 1.1. *The market* (1.2) *admits a martingale probability measure if and only if the functions $a(t), b(t)$ are such that*

$$e^d < 1 + a(t) - b(t)t, \quad \textit{and} \quad 1 + a(t) + b(t)t < e^u, \tag{1.7}$$

for all $t = 0, 1, \dots, N-1$. Moreover, when it exists, the martingale probability measure is unique and it is given by $p_t(k) = q_t(k)$, where

$$q_t(k) = \frac{1 + a(t-1) + b(t-1)k - e^d}{e^u - e^d}, \tag{1.8}$$

where $t = 1, \dots, N, k \in \mathrm{Im}(M_{t-1}) = \{-t+1, -t+3, \dots, t-3, t-1\}$. Thus, under the conditions (1.7), the market (1.2) is complete.

Proof. As $\{S^*(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$, it suffices to prove that

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = S^*(t-1), \quad t = 1, 2, \dots, N. \tag{1.9}$$

As $R(t)$ is measurable with respect to M_t, the discount process can be taken out from the conditional expectation in the left hand side of (1.9), hence

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = \frac{\mathbb{E}[S(t)|M_0, \dots, M_{t-1}]}{(1+R(0))\dots(1+R(t-1))} = \frac{\mathbb{E}[S(t)|M_{t-1}]}{(1+R(0))\dots(1+R(t-1))},$$

where for the second equality we use that $\{S(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$ and that $\{M_t\}_{t=0,\dots,N}$ is a Markov process (see Remark 0.1). Writing $S(t) = \frac{S(t)}{S(t-1)} S(t-1)$ and using that $S(t-1)$ is M_{t-1}-measurable we obtain

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = \frac{S(t-1)}{(1+R(0))\dots(1+R(t-1))} \mathbb{E}[\frac{S(t)}{S(t-1)}|M_{t-1}].$$

Next we use

$$\frac{S(t-1)}{(1+R(0))\dots(1+R(t-1))} = \frac{S^*(t-1)}{1+R(t-1)}, \quad \frac{S(t)}{S(t-1)} = e^{\frac{u+d}{2}} e^{\frac{u-d}{2}(M_t - M_{t-1})}.$$

According to (1.1), the increments of the process $\{M_t\}_{t=0,\dots,N}$ satisfy

$$\mathbb{P}(M_t - M_{t-1} = 1|M_{t-1} = k) = p_t(k), \quad \mathbb{P}(M_t - M_{t-1} = -1|M_{t-1} = k) = 1 - p_t(k).$$

Hence

$$\begin{aligned}\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] &= \frac{S^*(t-1)}{1+R(t-1)} \mathbb{E}[e^{\frac{u+d}{2}} e^{\frac{u-d}{2}(M_t - M_{t-1})}|M_{t-1}] \\ &= S^*(t-1) \frac{e^{\frac{u+d}{2}}}{1 + a(t-1) + b(t-1)k} \left(e^{\frac{u-d}{2}} p_t(k) + e^{-\frac{u-d}{2}} (1 - p_t(k)) \right).\end{aligned}$$

Thus in order for $p_t(k)$ to be a martingale probability it must hold that

$$\frac{e^{\frac{u+d}{2}}}{1 + a(t-1) + b(t-1)k} \left(e^{\frac{u-d}{2}} p_t(k) + e^{-\frac{u-d}{2}} (1 - p_t(k)) \right) = 1.$$

Solving the latter equation we find $p_t(k) = q_t(k)$, where $q_t(k)$ is given by (1.8). Moreover $0 < q_t(k) < 1$ holds if and only if (1.7) are satisfied, which concludes the proof of the theorem. □

Remark 1.2. It is clear that the transition probabilities are constant if and only if $b \equiv 0$ and $a(t) = a(0)$, for all $t = 1, \dots, N-1$, i.e., if and only if the risk-free rate is a deterministic constant, in which case we go back to the standard binomial model.

Example. Let a_0, b_0 be constants such that $b_0 > 0$ and $a_0 > b_0 - 1$. When

$$a(t) = a(0) := a_0, \quad b(t) = \frac{b_0}{t}, \quad t = 1, \dots, N-1, \tag{1.10}$$

the conditions (1.7) become

$$e^d < 1 + a_0 - b_0, \quad e^u > 1 + a_0 + b_0$$

and the martingale transition probabilities read

$$q_1(0) = \frac{1 + a_0 - e^d}{e^u - e^d}, \quad q_t(k) = \frac{1 + a_0 + \frac{b_0 k}{t-1} - e^d}{e^u - e^d},$$

for $t = 2, \dots, N$, $k \in \{-t+1, -t+3, \dots, t-1\}$. We shall use this example later on.

European derivatives on the stock

Next we study the problem of pricing European derivatives in the market (1.2).

Definition 1.1. *Assume that the market* (1.2) *is complete (e.g., the risk-free rate is given by the Ho-Lee model and the conditions* (1.7) *are verified). Consider a European derivative with maturity $T = N$ and pay-off Y which is measurable with respect to $M_0, \dots, M_N$ (e.g., $Y = g(S(N))$ for a standard European derivative on the stock). The risk-neutral price of the derivative is given by*

$$\Pi_Y(t) = D(t)^{-1} \widetilde{\mathbb{E}}[D(T)Y | M_0, \dots, M_t], \quad t = 0, \dots, T, \tag{1.11}$$

where $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation in the martingale probability measure. In particular, $\Pi_Y(0) = \widetilde{\mathbb{E}}[D(T)Y]$ and $\Pi_Y(T) = Y$.

For example, the **zero coupon bond** (ZCB) with **face value** K and maturity T is the European style derivative that promises to pay K at time T. It follows by Definition 1.11 that the value of the ZCB at time t is given by

$$B_K(t, T) = K D(t)^{-1} \widetilde{\mathbb{E}}[D(T) | M_0, \dots, M_t] \quad t = 0, \dots, T = N.$$

When $K = 1$ we denote $B_K(t, T)$ simply as $B(t, T)$, that is

$$B(t, T) = D(t)^{-1} \widetilde{\mathbb{E}}[D(T) | M_0, \dots, M_t], \quad t = 0, \dots, T. \tag{1.12}$$

Clearly, $B_K(t, T) = K B(t, T)$. Moreover the following variant of the put-call parity holds in the market (1.2):

$$\Pi_{\text{call}}(t) - \Pi_{\text{put}}(t) = S(t) - K B(t, T). \tag{1.13}$$

Task 1.1 (*). *Prove* (1.13).

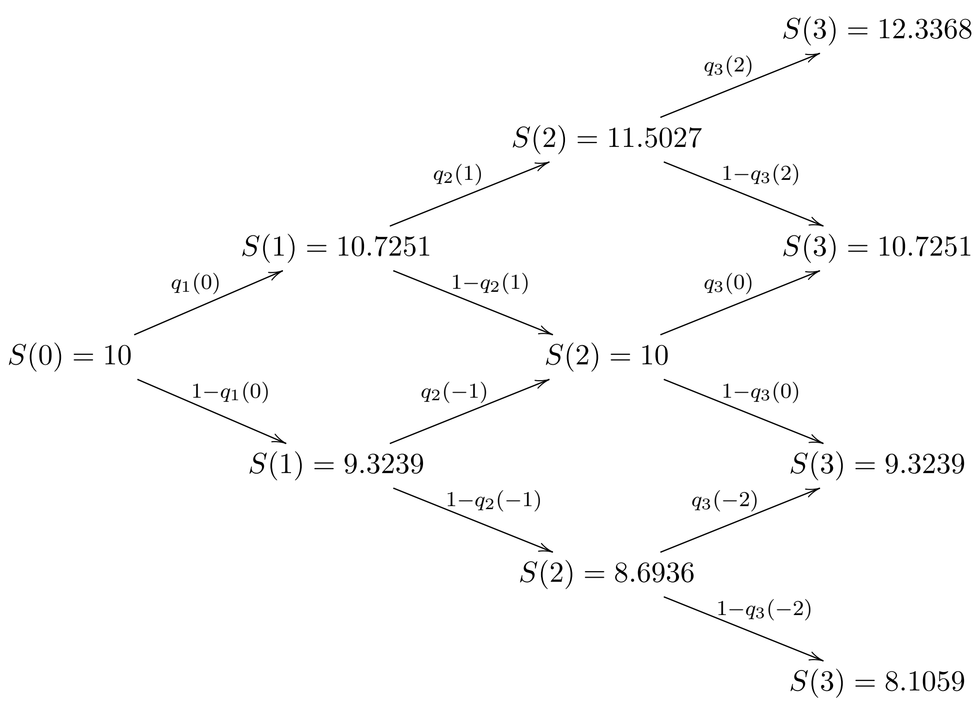
Example in the 3-period model with Ho-Lee risk-free rate. Consider a binomial stock price with $N = 3$, $u = -d = 0.07$, $S_0 = 10$ and a Ho-Lee model for the interest rate with parameters

$$a(0) = R_0 = 0.03, \quad a(1) = 0.05, \quad a(2) = 0.04, \quad b(1) = 0.02, \quad b(2) = 0.01.$$

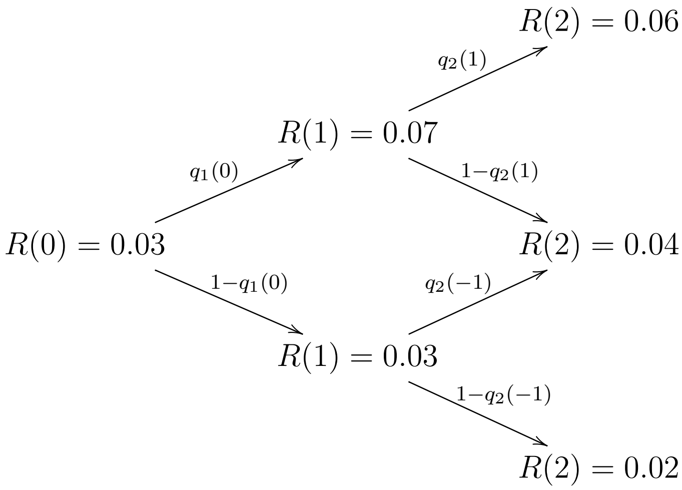
The martingale transition probabilities are

$$q_1(0) = \frac{1 + R_0 - e^d}{e^u - e^d} = 0.6966$$
$$q_2(1) = \frac{1 + a(1) + b(1) - e^d}{e^u - e^d} = 0.9821$$
$$q_2(-1) = \frac{1 + a(1) - b(1) - e^d}{e^u - e^d} = 0.6966$$
$$q_3(2) = \frac{1 + a(2) + 2b(2) - e^d}{e^u - e^d} = 0.9107$$
$$q_3(0) = \frac{1 + a(2) - e^d}{e^u - e^d} = 0.7680$$
$$q_3(-2) = \frac{1 + a(2) - 2b(2) - e^d}{e^u - e^d} = 0.6252$$

As $q_t(k) \in (0, 1)$, the market is complete. The binomial tree for the stock price in the martingale probability is as follows



The binomial tree for the interest rate is



The discount process in the martingale probability has the following distribution

$$D(0) = 1, \quad D(1) = \frac{1}{1+R(0)} = 0.9709, \quad \text{with prob. } 1,$$

$$D(2) = \frac{D(1)}{1+R(1)} = \begin{cases} \frac{D(1)}{1+0.07} = 0.9074, & \text{with prob. } q_1(0) \\ \frac{D(1)}{1+0.03} = 0.9426 & \text{with prob. } 1-q_1(0) \end{cases},$$

$$D(3) = \frac{D(2)}{1+R(2)} = \begin{cases} \frac{0.9074}{1+0.06} = 0.8560, & \text{with prob. } q_1(0)q_2(1) \\ \frac{0.9074}{1+0.04} = 0.8725 & \text{with prob. } q_1(1)(1-q_2(1)) \\ \frac{0.9426}{1+0.04} = 0.9063 & \text{with prob. } (1-q_1(0))q_2(-1) \\ \frac{0.9426}{1+0.02} = 0.9241 & \text{with prob. } (1-q_1(0))(1-q_2(-1)) \end{cases}$$

Now assume that we want to compute the initial price of a call option on the stock with strike $K = 10$ and maturity $T = 3$. According to Definition 1.1, this price is given by

$$\Pi(0) = \widetilde{\mathbb{E}}[D(3)(S(3)-10)_+],$$

where the expectation is in the martingale probability $q_t(k)$. To compute this expectation we need the joint distribution of the random variables $D(3), S(3)$. Using our results above we find that this joint distribution is given as in the following table:

$D(3)\downarrow$, $S(3)\rightarrow$	12.3368	10.7251	9.3239	8.1059
0.8560	$q_1(0)q_2(1)q_3(2)$	$q_1(0)q_2(1)(1-q_3(2))$	0	0
0.8725	0	$q_1(0)(1-q_2(1))q_3(0)$	$q_1(0)(1-q_2(1))(1-q_3(0))$	0
0.9063	0	$(1-q_1(0))q_2(-1)q_3(0)$	$(1-q_1(0))q_2(-1)(1-q_3(0))$	0
0.9241	0	0	$(1-q_1(0))(1-q_2(-1))q_3(-2)$	$(1-q_1(0))(1-q_2(-1))(1-q_3(-2))$

We conclude that

$$\begin{aligned}\Pi(0) = {} & 0.8560[(12.3368-10)q_1(0)q_2(1)q_3(2) + (10.7251-10)q_1(0)q_2(1)(1-q_3(2))] \\ & + 0.8725(10.7251-10)q_1(0)(1-q_2(1))q_3(0) + 0.9063(1-q_1(0))q_2(-1)q_3(0) = 1.4373.\end{aligned}$$

Task 1.2 (*). *In the example just considered compute the possible prices of the call at times $t = 1, 2$.*

For future purpose we also compute the initial price of the ZCB expiring at time $T = 3$. According to (1.12), the value at time $t = 0$ of the ZCB with face value 1 is given by $B(0,T) = \widetilde{\mathbb{E}}[D(T)]$. Hence

$$B(0,3) = \widetilde{\mathbb{E}}[D(3)] = 0.8560 q_1(0) q_2(1) + 0.8725 q_1(0)(1 - q_2(1)) + 0.9063(1 - q_1(0)) q_2(-1) + 0.9241(1 - q_1(0))(1 - q_2(-1)) = 0.8731. \tag{1.14}$$

1.2 Forward and Futures

Forward contracts

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a European type financial derivative stipulated by two parties in which one promises to the other to sell (and possibly deliver) the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to option contracts, both parties in a forward contract are *obliged* to fulfill their part of the agreement. In particular, as they both have the same right/obligation, neither of the two parties has to pay a premium to the other when the contract is stipulated, that is to say, *forward contracts are free*; in fact, the terminology used for forward contracts is "to enter a forward contract" and not "to buy/sell a forward contract". The party who must sell the asset at maturity holds the short position, while the party who must buy the asset is the holder of the long position. Hence the pay-off for a long position in a forward contract on the asset $\mathcal{U}$ is

$$Y_{\text{long}} = (\Pi^{\mathcal{U}}(T) - K),$$

while for the holder of the short position the pay-off is

$$Y_{\text{short}} = (K - \Pi^{\mathcal{U}}(T)).$$

Forward contracts are traded OTC and most commonly on commodities or market indexes, such as currency exchange rates, interest rates and volatilities. In the case that the underlying asset is an index, forward contracts are also called **swaps** (e.g., currency swaps, interest rate swaps, volatility swaps, etc.).

One purpose of forward contracts is to share risks. Irrespective of the movement of the underlying asset in the market, its price at time T for the holders of the forward contract will be K. The delivery price agreed by the two parties in a forward contract is also called the **forward price** of the asset. More precisely, the T-forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ of an asset $\mathcal{U}$ at time $t < T$ is the strike price of a forward contract on $\mathcal{U}$ stipulated at time t and with maturity T, while the current, actual price $\Pi^{\mathcal{U}}(t)$ of the asset is called the **spot price**. Note that the forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ is unlikely to be a good estimation for the price of the asset at time T, since the consensus on this value is limited to the participants of the forward contract and different parties may agree to different delivery prices. The delivery price of futures contracts on the asset, which we define in the next section, gives a better and more commonly accepted estimation for the future value of an asset.

Theorem 1.2. *Suppose that the market $\{\Pi^{\mathcal{U}}(t), B(t)\}_{t=0,\dots,N}$ is given by* (1.2) *(with $S(t) = \Pi^{\mathcal{U}}(t)$) and that the market is complete. The forward price of the asset $\mathcal{U}$ for delivery at time $T = N$ is given by*

$$\mathrm{For}_{\mathcal{U}}(t,T) = \frac{\Pi^{\mathcal{U}}(t)}{B(t,T)}, \tag{1.15}$$

where $B(t,T)$ is given by (1.12)*, that is to say, $B(t,T)$ is the risk-neutral price at time t of the ZCB with face value 1 and expiring at time T.*

Proof. The proof is very simple. First we remark that having a long position on a forward contract with maturity T and delivery price K is equivalent to hold a portfolio which is long 1 share of the call on the asset and short 1 share of the put on the asset, both options with maturity T and strike price K. Indeed, irrespective of the price of the asset at time T, this portfolio entails that we will buy the asset at time T for the price K. Hence if we denote by $F(t), C(t), P(t)$ the arbitrage-free value at time $t < T$ of the forward contract, the call option and the put option respectively, then it must hold

$$C(t) - P(t) = F(t).$$

On the other hand, by the put-call parity (1.13),

$$C(t) - P(t) = \Pi(t) - KB(t,T).$$

As forward contracts have zero value, then $F(t) = 0$ must hold, which is the case if and only if $K = \Pi(t)/B(t,T)$. By definition of forward price, this completes the proof of the theorem. □

For instance, in the 3-period market model considered in Section 1.1, we have found $B(0,3) = 0.8731$, see (1.14). Hence the 3-forward price at time $t = 0$ of the stock in that market is $\mathrm{For}(0,T) = S(0)/B(0,3) = 10/0.8731 = 11.4534$.

Futures

Futures are standardized forward contracts, i.e., rather than being OTC, they are negotiated in regularized markets. Specifically, a **futures market** is a market in which the object of trading are futures contracts. Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation, and so in particular all contracts on the same asset with the same time of maturity T have the same delivery price. The T-**future price**

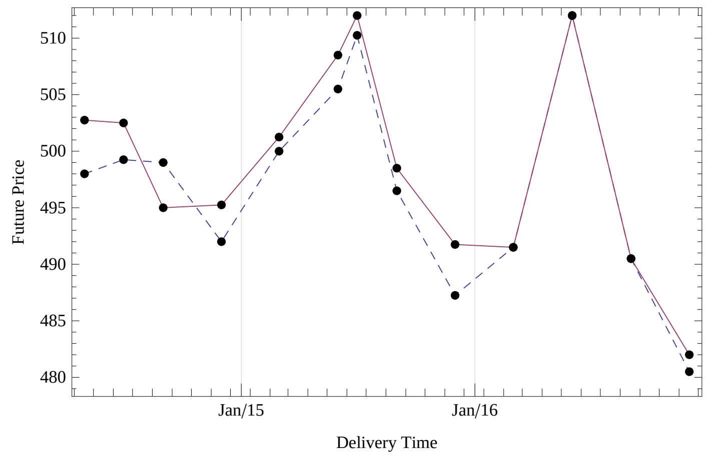


Figure 1.1: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

$\mathrm{Fut}_{\mathcal{U}}(t,T)$ of the asset $\mathcal{U}$ at time $t \leq T$ is defined as the delivery price at time $t \leq T$ in the futures contract on the asset $\mathcal{U}$ with maturity T. Futures markets have been existing for more than 300 years and nowadays the most important ones are the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE).

In a futures market, anyone (after a proper authorization) can stipulate a futures contract. More precisely, holding a position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held. The cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down. Moreover, in order to alleviate the risk of insolvency, the cash flow is distributed in time through the mechanism of the **margin account**. For example, assume that at $t = 0$ we open a long position in a futures contract expiring at time T. At the same time, we need to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)$ will be added to the account, if it positive, or withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$(\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(t-1,T)) + (\mathrm{Fut}_{\mathcal{U}}(t-1,T) - \mathrm{Fut}_{\mathcal{U}}(t-2,T)) + \\ \cdots + (\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)) = (\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)).$$

If a long position is held up to the time of maturity, then the holder of the long position should buy the underlying asset. However futures contracts are often **cash settled** and not **physically settled**, which means that the delivery of the underlying asset does not occur,

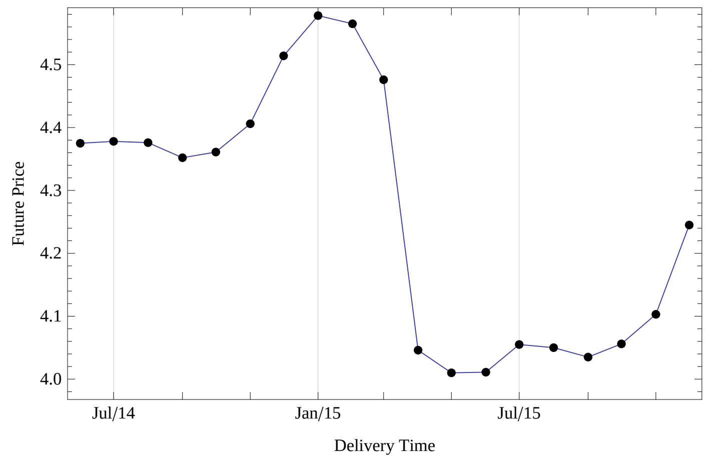


Figure 1.2: Futures price of natural gas on May 13, 2014 for different delivery times

and the equivalent value in cash is paid instead.

Our next purpose is to introduce the definition of arbitrage-free future price of an asset. Our strategy is to show is that any reasonable definition should satisfy 3 standard conditions and then show that these conditions define uniquely the future price as a stochastic process.

For simplicity we argue under the assumption that the underlying asset $\mathcal{U}$ and the money market make up a complete market of the form (1.2) with $S(t) = \Pi^{\mathcal{U}}(t)$ and $T = N$. As the generalized random walk $\{M_t\}_{t=0,\dots,N}$ contains all the information about the state of the market, we are naturally led to impose the following first condition on the future price.

Assumption 1. The future price process $\{\mathrm{Fut}_{\mathcal{U}}(t,T)\}_{t=0,\dots,T=N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$.

For the next assumption we need to define the concept of self-financing portfolio process invested in the futures contract and the money market. Consider a portfolio process that, at time $t < T$, consists of $h(t)$ shares of the futures contract expiring at time T and $h_{t+1}(t)$ shares of the ZCB maturing at time $t+1$. As the ZCB has very short time left to maturity, then $h_t(t+1)$ is our position on the money market (recall that the money market consists of short term loan assets). We assume that the portfolio process is predictable from $\{M_t\}_{t=0,\dots,N}$. As futures contracts have zero value, the value of the portfolio at time t is simply the money market account:

$$V(t) = h_{t+1}(t)B(t,t+1) = \frac{h_{t+1}(t)}{1+R(t)}.$$

At time $t+1$ the portfolio generates the cash flow

$$\begin{aligned} C(t+1) &= h_{t+1}(t) + h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) \\ &= V(t)(1+R(t)) + h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)). \end{aligned}$$

In a self-financing portfolio this cash should be immediately re-invested in the money market[1]. Hence $C(t+1) = h_{t+2}(t+1)B(t+1,t+2) = V(t+1)$. It follows that

$$h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) = V(t+1) - (1+R(t))V(t) = V(t+1) - \frac{D(t)}{D(t+1)}V(t)$$
$$= D(t+1)^{-1}[D(t+1)V(t+1) - D(t)V(t)].$$

Hence

$$h(t)D(t+1)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) = V^*(t+1) - V^*(t). \tag{1.16}$$

Definition 1.2. *A predictable portfolio process invested in the futures contract and the money market is said to be self-financing if its discounted value satisfies* (1.16).

Now, by the arbitrage-free principle, any self-financing portfolio invested in futures and in the money market should not be an arbitrage. We have seen that this condition can be achieved by imposing that the discounted value of predictable self-financing portfolio processes is a martingale. This holds in particular if

$$\widetilde{\mathbb{E}}[V^*(t+1)|M_0,\dots,M_t] = V^*(t),$$

for all $t = 0,\dots,T-1$. Hence, taking the conditional expectation of both sides of (1.16) with respect to $M_0,\dots,M_t$, we obtain

$$h(t)D(t+1)\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[V^*(t+1) - V^*(t)|M_0,\dots,M_t] = 0,$$

where we used that $h(t)$ and $D(t+1)$ are measurable with respect to $M_0,\dots,M_t$ and thus can be taken out from the conditional expectation. By Assumption 1 we have

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T)|M_0,\dots,M_t] - \mathrm{Fut}_{\mathcal{U}}(t,T).$$

Hence the market is free of self-financing arbitrages if we assume the following.

Assumption 2. The future price satisfies

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T)|M_0,\dots,M_t] = \mathrm{Fut}_{\mathcal{U}}(t,T), \quad t = 0,\dots,T-1.$$

The last natural assumption is that the future price at maturity $t = T$ should coincide with the spot price $\Pi(T)$ of the asset, i.e.,

Assumption 3. $\mathrm{Fut}_{\mathcal{U}}(T,T) = \Pi(T)$.

Theorem 1.3. *There is only one stochastic process* $\{\mathrm{Fut}_{\mathcal{U}}(t,T)\}_{t=0,\dots,T}$ *that satisfies Assumptions 1-3, namely*

$$\mathrm{Fut}_{\mathcal{U}}(t,T) = \widetilde{\mathbb{E}}[\Pi(T)|M_0,\dots,M_t]. \tag{1.17}$$

[1] This is the only possibility, as changing position on the futures contract costs nothing.

Proof. The proof that (1.17) satisfies 1-3 follows easily by the properties of the conditional expectation. Now, Assumptions 2-3 imply directly that (1.17) holds at time $t = T - 1$, for

$$\mathrm{Fut}_{\mathcal{U}}(T-1,T) = \widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(T,T)|M_0,\dots,M_{T-1}] = \widetilde{\mathbb{E}}[\Pi(\mathrm{T})|M_0,\dots,M_{T-1}],$$

where we used Assumption 2 with $t = T - 1$ in the first equality, and assumption 3 in the second equality. Now the proof of the theorem can be easily completed by induction. $\square$

As an example, consider again the 3-period model in Section 1.1. The 3-future price at time $t = 0$ of the stock in that market is

$$\begin{aligned}\mathrm{Fut}(0,3) = \widetilde{\mathbb{E}}[S(3)] &= q_1(0)q_2(1)q_3(2)12.3368\\ &+ [q_1(0)q_2(1)(1-q_3(2)) + q_1(0)(1-q_2(1))q_3(0) + (1-q_1(0))q_2(-1)q_3(0)]10.7251\\ &+ [q_1(0)(1-q_2(1))(1-q_3(0)) + (1-q_1(0))q_2(-1)(1-q_3(0))\\ &+ (1-q_1(0))(1-q_2(-1))q_3(-2)]9.3239\\ &+ (1-q_1(0))(1-q_2(-1))(1-q_3(-2))6.1059 = 11.6039\end{aligned}$$

Recall that the 3-forward price at time $t = 0$ of the same stock is 11.4534, which was computed at the end of Section 1.2. The general relation between forward and future price of an asset is given in the following theorem.

Theorem 1.4. *The* **Forward-Future spread** *of an asset, i.e., the difference between its forward and future price, satisfies*

$$\mathrm{For}_{\mathcal{U}}(0,T) - \mathrm{Fut}_{\mathcal{U}}(0,T) = \frac{\widetilde{\mathrm{Cov}}[D(T),\Pi(T)]}{\widetilde{\mathbb{E}}[D(T)]},$$

where $\widetilde{\mathrm{Cov}}$ *is the covariance in the risk-neutral probability. Moreover if the interest rate of the money market is deterministic then* $\mathrm{Fut}_{\mathcal{U}}(0,T) = \mathrm{For}(0,T)$.

Task 1.3 (*)**.** *Prove the theorem.*

Task 1.4 (Matlab)**.** *Write a Matlab code that compute the future price at time $t = 0$ of an asset in the market* (1.2)*, where the risk-free rate follows the Ho-Lee model with functions $a(t), b(t)$ given by* (1.10)*. Study how the future price curve $T \to \mathrm{Fut}_{\mathcal{U}}(0,T)$ depend on the parameters a_0, b_0. Hint: Letting $x = \{-1,1\}^N$ be a possible path for the price of the underlying asset, we have*

$$\mathrm{Fut}_{\mathcal{U}}(0,T) = \widetilde{\mathbb{E}}[\Pi^{\mathcal{U}}(T)] = \sum_{x\in\{-1,1\}^N} \widetilde{\mathbb{P}}(x)\Pi^{\mathcal{U}}(T,x) \tag{1.18}$$

where

$$\Pi^{\mathcal{U}}(T,x) = \Pi^{\mathcal{U}}(0)\exp\left(T\frac{u+d}{2} + \frac{u-d}{2}(x_1+\cdots+x_T)\right)$$

is the asset price at time T along the path x and $\widetilde{\mathbb{P}}(x)$ is the risk-neutral probability of realization of the path x, which is computed according to (5)*. To apply* (1.18) *you need to create all 2^N elements $x \in \{-1,1\}^N$, which is possible only for a relatively small number of steps (up to, say, $N \approx 20$).*

Chapter 2

A project on the trinomial model

As opposed to the binomial options pricing model, the trinomial model is an **incomplete** model, that is to say, the risk-neutral price of European derivatives in a trinomial market is not uniquely defined. Some scholars believe that real markets are incomplete, due to the fact that investors assign different values to the market price of risk (i.e., choose a different risk-neutral probability to price European derivatives). In this project the trinomial model is studied in details, in particular regarding the problem of pricing and hedging European derivatives by "almost" self-financing and hedging portfolios.

2.1 The trinomial model

In the trinomial model the stock price is allowed to move in three different directions at each time step, namely $S(0) = S_0 > 0$ and

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with prob. } p_u \\ S(t-1)e^{m} & \text{with prob. } p_m \\ S(t-1)e^{d} & \text{with prob. } p_d \end{cases} \qquad t = 1, \dots, N,$$

where $u > m > d$, $0 < p_u, p_m, p_d < 1$ and $p_u + p_m + p_d = 1$. The risk-free asset has value $B(t) = B_0 e^{rt}$, where r is constant.

The possible prices of the stock at time $t = 0, \dots, N$ satisfy

$$S(t) \in \{S_0 e^{N_u(t)u + N_d(t)d + (t - N_u(t) - N_d(t))m} \ \text{ for } N_u(t), N_d(t) = 0, \dots, t \text{ and } N_u(t) + N_d(t) \leq t\}.$$

It follows that the number of possible stock prices at time t is

$$\sum_{N_u=0}^{t} \sum_{N_d=0}^{t-N_u} 1 = \sum_{N_u=0}^{t} (t - N_u + 1) = (t+1)t + t + 1 - \sum_{N_u=0}^{t} N_u$$
$$= (t+1)t + t + 1 - \frac{(t+1)t}{2} = \frac{(t+1)(t+2)}{2}.$$

Thus the number of nodes in the trinomial tree grows quadratically—while we recall that for the binomial model this grow was linear ($t+1$ possible prices at time t). To reduce the number of nodes in the trinomial tree we shall assume from now on that the recombination condition holds:

$$m = \frac{u+d}{2}$$

and thus restrict the trinomial stock price to the form

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with prob. } p_u \\ S(t-1)e^{\frac{u+d}{2}} & \text{with prob. } p_m \\ S(t-1)e^{d} & \text{with prob. } p_u \end{cases} \qquad t = 1, \dots, N, \tag{2.1}$$

with $u > d$. In this case the possible stock prices at time t belong to the set

$$\{S_0 e^{(u-d)(N_u(t)-N_d(t))/2+(u+d)t/2)}, N_u(t), N_d(t) = 0, \dots, t\},$$

which contains $2t+1$ elements. Hence the number of nodes of the trinomial tree with the recombination condition grows linearly, as for the binomial model. In the following we restrict to this case for simplicity.

Probabilistic formulation. Let $\Omega = \{-1, 0, 1\}^N$. Given $p = (p_u, p_m, p_d)$ such that $0 < p_u, p_m, p_d < 1$ and $p_u + p_m + p_d = 1$, we define the probability $\mathbb{P}_p$ on the sample space Ω by letting

$$\mathbb{P}_p(\omega) = p_u^{N_+(\omega)} p_m^{N_0(\omega)} p_d^{N_-(\omega)},$$

where $N_\pm(\omega)$ is the number of ± 1 in the sample ω and $N_0(\omega) = N - N_+(\omega) - N_-(\omega)$ the number of 0's. The trinomial stock price can be regarded as a stochastic process in the probability space $(\Omega, \mathbb{P}_p)$. To see this let the stochastic process $\{X_t\}_{t=1,\dots,N}$ be defined on $\omega = (\gamma_1, \dots, \gamma_N) \in \Omega$ as $X(\omega) = \gamma_t$, that is

$$X_t(\omega) = \begin{cases} -1 & \text{if } \gamma_t = -1 \\ 0 & \text{if } \gamma_t = 0 \\ 1 & \text{if } \gamma_t = 1 \end{cases} . \tag{2.2}$$

Note that the random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d.**). We can write (2.1) as

$$S(t) = S(t-1) \exp\left[\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) X_t\right]. \tag{2.3}$$

Iterating the previous identity, the trinomial stock price at time $t = 1, \dots, N$ is

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) Z_t\right], \quad Z_t = X_1 + \dots + X_t. \tag{2.4}$$

Hence $S(t) : \Omega \to \mathbb{R}$ and $\{S(t)\}_{t=0,\dots,N}$ is a stochastic process on the probability space $(\Omega, \mathbb{P}_p)$. Letting $Z_0 = 0$, the process $\{S(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{Z_t\}_{t=0,\dots,N}$. Moreover we have the following analogue of Theorem 0.4.

Theorem 2.1. *The probability measure $\mathbb{P}_p$ is a martingale measure if and only if $p = q = (q_u, q_m, q_d)$, where (q_u, q_m, q_d) satisfy*

$$q_u e^u + q_m e^{\frac{u+d}{2}} + q_d e^d = e^r, \tag{2.5a}$$
$$q_u + q_m + q_d = 1, \tag{2.5b}$$
$$0 < q_u, q_m, q_d < 1. \tag{2.5c}$$

Task 2.1 (*). *Prove the theorem.*

We remark that there exists infinitely many triples that satisfy (2.5). Indeed the solution of (2.5a)-(2.5b) can be written in parametric form as

$$q_u = \frac{e^r - e^d}{e^u - e^d} - \omega \frac{e^{d/2}}{e^{u/2} + e^{d/2}}, \quad q_m = \omega, \quad q_d = \frac{e^u - e^r}{e^u - e^d} - \omega \frac{e^{u/2}}{e^{u/2} + e^{d/2}} \tag{2.6}$$

and, under suitable conditions on the market parameters r, u, d and the free parameter ω, all such solutions define a probability, i.e., they satisfy (2.5c). Note also that in the limit $\omega \to 0$ the trinomial model reduces to the binomial model and the solutions (2.6) converge to the martingale probability measure of the binomial model.

Task 2.2 (*). *Let $r > 0$, $u > 0$ and $u = -d$. Show that the triples* (2.6) *satisfy* (2.5c) *if and only if*

$$u > r \quad \text{and} \quad 0 < \omega < \frac{e^u - e^r}{e^u - 1}.$$

The existence of a martingale probability measure ensures that the trinomial market is free of self-financing arbitrages, see Remark 0.3. However the non-uniqueness of such measure prevents to fix uniquely the price of European derivatives. Some practitioners have a positive view of this property of the trinomial model, since the freedom in choosing the parameter ω can be used to better calibrate the model. However, regardless of which martingale measure one chooses, it is generally not possible to hedge European derivatives self-financially, that is to say, the trinomial model is incomplete (see Remark 0.5). To see this, consider a one-period model with $u = -d$ and a derivative with pay off $Y = g(S(1))$. A (constant) portfolio (h_S, h_B) hedging the derivative should satisfy $h_S S(1) + h_B B_0 e^{rt} = g(S(1))$ for all possible values of $S(1)$. This leads to the three equations

$$h_S S_0 e^u + h_B B_0 e^{rt} = g(S_0 e^u),$$
$$h_S S_0 + h_B B_0 e^{rt} = g(S_0),$$
$$h_S S_0 e^{-u} + h_B B_0 e^{rt} = g(S_0 e^{-u}).$$

This system has a solution (h_S, h_B) if and only if

$$g(S_0 e^{-u}) - g(S_0) e^{-u} - g(S_0) + g(S_0 e^u) e^{-u} = 0,$$

which is satisfied only for very particular choices of the pay-off function and of the market parameters. For instance for a call option with strike $K = S_0$, the latter equation becomes

$$(e^{-u} - 1)_+ + (e^u - 1)_+ e^{-u} = 0,$$

which has the only solution $u = 0$.

Incomplete models, of which the trinomial model is just an example, are investigated extensively by scholars and the community is divided among those who believe that incomplete models should be rejected and others who instead believe that real markets are incomplete and therefore incomplete models are more realistic.

2.2 Pricing and hedging in incomplete markets

The most common approach to incompleteness is to accept it as an attribute of real markets. That is to say, in real markets there is no only one acceptable fair price for financial derivatives and moreover financial derivatives cannot be perfectly hedged by self-financing portfolios. Let's discuss how the trinomial model can address these two properties.

Pricing

In the incomplete trinomial model there exist infinitely many martingale measures (q_u, q_m, q_d), see (2.6). Each martingale measure gives rise to a different price for the European derivative with pay-off Y at maturity $T = N$; denoting by $\mathbb{E}_\omega$ the expectation in the probability measure (2.6) and by $\Pi_Y(t, \omega)$ the price of the derivative derived from this measure, we have

$$\Pi_Y(t, \omega) = e^{-r(N-t)} \mathbb{E}_\omega [Y | S(1), \dots, S(t)].$$

Task 2.3 (*). *Prove the recurrence formula* $\Pi_Y(N, \omega) = Y$,

$$\Pi_Y(t, \omega) = e^{-r}[q_u \Pi_Y^u(t+1, \omega) + q_m \Pi_Y^m(t+1, \omega) + q_d \Pi_Y^d(t+1, \omega)], \quad t = 0, \dots, N-1. \tag{2.7}$$

In Task 2.4 below it is asked to compute $\Pi_Y(0, \omega)$ with Matlab using the recurrence formula (2.7). To simplify the analysis we assume that the parameters of the trinomial model are

$$u = -d, \quad 0 < r < u, \quad p_u = p_d = p \in (0, 1/2). \tag{2.8}$$

Thus (2.3) becomes

$$S(t) = S(t-1)e^{uX_t}, \quad X_t = \begin{cases} -1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-2p \\ 1 & \text{with prob. } p \end{cases}. \tag{2.9}$$

Moreover, according to Task 2.2, for each value

$$0 < \omega < \frac{e^u - e^r}{e^u - 1} := \omega_{\max}(r, u),$$

we have the martingale probability defined by

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} - \omega \frac{e^{-u/2}}{e^{u/2} + e^{-u/2}}, \quad q_m = \omega, \quad q_d = \frac{e^u - e^r}{e^u - e^{-u}} - \omega \frac{e^{u/2}}{e^{u/2} + e^{-u/2}}. \tag{2.10}$$

Now let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$. Define $S(0) = S_0$ and

$$S(t_i) = S(t_{i-1})e^{uX_i}, \quad i = 1, \dots, N, \tag{2.11}$$

where the random variables $X_1, \dots, X_N$ are given by (2.9). The instantaneous variance of the stock is defined, as for the binomial model, by

$$\sigma^2 = \frac{1}{h}\mathrm{Var}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{2}{h} p u^2. \tag{2.12}$$

Having chosen $u = -d$, the instantaneous mean of log return is zero. The interest rate on each period becomes rh and, according to (2.12), $u = \sqrt{\frac{h}{2p}}\sigma$. It is easy to see that

$$\omega_{\max}\Big(rh, \sqrt{\frac{h}{2p}}\sigma\Big) \to 1, \text{ as } h \to 0^+.$$

Hence provided h is sufficiently small we can assume that $0 \leq \omega \lesssim 1$. Moreover the recurrence formula (2.7) becomes $\Pi_Y(t_N, \omega) = Y$, and

$$\Pi_Y(t_i, \omega) = e^{-rh}[q_u \Pi_Y^u(t_{i+1}, \omega) + q_m \Pi_Y^m(t_{i+1}, \omega) + q_d \Pi_Y^d(t_{i+1}, \omega)], \quad i = 0, \dots, N-1. \tag{2.13}$$

Task 2.4 (Matlab). *Write a Matlab function*

$$\texttt{EuroZeroTrin}(\texttt{g}, \texttt{T}, \texttt{s}, \texttt{sigma}, \texttt{r}, \texttt{p}, \texttt{omega}, \texttt{N})$$

that computes the trinomial price at time $t = 0$ of the standard European derivative with pay-off $Y = g(S(T))$ when $\omega \in (0, 1)$ is fixed. Show numerically that the result depends on the probability p. Plot the curves $\omega \to \Pi_Y(0, \omega)$ for different values of p and show numerically that the binomial and the trinomial price converge to the same value as $N \to \infty$ only for $\omega = 1 - 2p$. For this value of ω, study the speed of convergence to the Black-Scholes price as $N \to \infty$ for different values of $p \in (0, 1/2)$ and look for the value p_* that gives the fastest convergence. Show that the trinomial model converges to the Black-Scholes price faster than the binomial model.

Hedging

European derivatives in the incomplete trinomial model cannot, in general, be hedged by self-financing portfolios. It is not hard to believe that this is often the case in real markets, which marks a point in favor of using incomplete models for real-world applications.

As hedging portfolios in incomplete markets cannot be, in general, self-financing, then we have to allow for cash-flows into the portfolio. Of course some restrictions in the cash flow are necessary, otherwise we could hedge the derivative by simply adding the cash required to pay-off the buyer just before the derivative expires. Here we discuss an approach for hedging in incomplete markets that can be seen as the "best approximation" to the usual self-financing strategy. For simplicity we restrict ourselves to the 2-period model.

The equations defining a self-financing hedging portfolio for the derivative with pay-off $Y = g(S(2))$ in a 2-period trinomial model are

$$\begin{aligned} &h_S(2)S(2) + h_B(2)B(2) = g(S(2)) \quad \text{(Hedging condition)} \\ &h_S(2)S(1) + h_B(2)B(1) = h_S(1)S(1) + h_B(1)B(1), \quad \text{(Self-financing condition)} \end{aligned}$$

We assume the values (2.8) for the market parameters. This implies in particular that the possible stock prices at time 1 are given by $S_0 e^{ju}$, for $j = -1, 0, 1$. We denote by $h_S(2, j)$ the portfolio position on the stock in the interval $(1, 2]$ assuming that the stock price at time 1 is $S_0 e^{ju}$, and with similar meaning we define $h_B(2, j)$. Hence the full portfolio process is described by 8 variables, namely

$$h_S(2,1),\ h_S(2,0),\ h_S(2,-1),\ h_B(2,1),\ h_B(2,0),\ h_B(2,-1), h_S(1),\ h_B(1).$$

The hedging/self-financing conditions are equivalent to the following 12 equations:

$$\left.\begin{aligned} &h_S(2,1)S_0e^{2u} + h_B(2,1)B_0e^{2r} = g(S_0e^{2u}) \\ &h_S(2,1)S_0e^{u} + h_B(2,1)B_0e^{2r} = g(S_0e^{u}) \\ &h_S(2,1)S_0 + h_B(2,1)B_0e^{2r} = g(S_0) \\ &h_S(2,0)S_0e^{u} + h_B(2,0)B_0e^{2r} = g(S_0e^{u}) \\ &h_S(2,0)S_0 + h_B(2,0)B_0e^{2r} = g(S_0) \\ &h_S(2,0)S_0e^{-u} + h_B(2,0)B_0e^{2r} = g(S_0e^{-u}) \\ &h_S(2,-1)S_0 + h_B(2,-1)B_0e^{2r} = g(S_0) \\ &h_S(2,-1)S_0e^{-u} + h_B(2,-1)B_0e^{2r} = g(S_0e^{-u}) \\ &h_S(2,-1)S_0e^{-2u} + h_B(2,-1)B_0e^{2r} = g(S_0e^{-2u}) \end{aligned}\right\} \quad \text{Hedging condition}$$

$$\left.\begin{aligned} &h_S(2,1)S_0e^{u} + h_B(2,1)B_0e^{r} = h_S(1)S_0e^{u} + h_B(1)B_0e^{r} \\ &h_S(2,0)S_0 + h_B(2,0)B_0e^{r} = h_S(1)S_0 + h_B(1)B_0e^{r} \\ &h_S(2,-1)S_0e^{-u} + h_B(2,-1)B_0e^{r} = h_S(1)S_0e^{-u} + h_B(1)B_0e^{r} \end{aligned}\right\} \quad \text{Self-financing condition}$$

or, in a more concise form,

$$h_S(2,j)S_0e^{(j+k)u} + h_B(2,j)e^{2r} = g(S_0e^{(j+k)u}), \tag{2.14}$$

$$h_S(2,j)S_0e^{ju} + h_B(2,j)e^{r} = h_S(1)e^{ju} + h_B(1)B_0e^{r} \quad \text{where} \quad j = -1,0,1, \quad k = -1,0,1. \tag{2.15}$$

The system of 9 equations for the hedging condition can be written in matrix from as $A\mathbf{x} = \mathbf{y}$ where

$$\mathbf{x} = \begin{pmatrix} h_S(2,1) \\ h_S(2,0) \\ h_S(2,-1) \\ h_B(2,1) \\ h_B(2,0) \\ h_B(2,-1) \\ h_S(1) \\ h_B(1) \end{pmatrix} \in \mathbb{R}^8, \qquad \mathbf{y} = \begin{pmatrix} g(S_0e^{2u}) \\ g(S_0e^u) \\ g(S_0) \\ g(S_0e^u) \\ g(S_0) \\ g(S_0e^{-u}) \\ g(S_0) \\ g(S_0e^{-u}) \\ g(S_0e^{-2u}) \end{pmatrix} \in \mathbb{R}^9$$

and A is the 9×8 matrix given by

$$A = \begin{pmatrix} S_0e^{2u} & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 & 0 \\ S_0e^u & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 & 0 \\ S_0 & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 & 0 \\ 0 & S_0e^u & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 \\ 0 & S_0 & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 \\ 0 & S_0e^{-u} & 0 & 0 & B_0e^{2r} & 0 & 0 & 0 \\ 0 & 0 & S_0 & 0 & 0 & B_0e^{2r} & 0 & 0 \\ 0 & 0 & S_0e^{-u} & 0 & 0 & B_0e^{2r} & 0 & 0 \\ 0 & 0 & S_0e^{-2u} & 0 & 0 & B_0e^{2r} & 0 & 0 \end{pmatrix}.$$

The system of 3 equations for the self-financing condition can be written in matrix form as $B\mathbf{x} = \mathbf{0}$, where B is the 3×8 matrix given by

$$B = \begin{pmatrix} S_0e^u & 0 & 0 & B_0e^r & 0 & 0 & -S_0e^u & -B_0e^r \\ 0 & S_0 & 0 & 0 & B_0e^r & 0 & -S_0 & -B_0e^r \\ 0 & 0 & S_0e^{-u} & 0 & 0 & B_0e^r & -S_0e^{-u} & -B_0e^r \end{pmatrix}.$$

Hence the full system on the hedging self-financing portfolio $\mathbf{x}$ is $G\mathbf{x} = \mathbf{b}$, where G is the 12×8 matrix and $\mathbf{b}$ is the 12×1 vector given by

$$G = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{y} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system $G\mathbf{x} = \mathbf{b}$ has (in general) no solutions, as there are more equations than unknowns. However, provided the 8×8 matrix G^TG is invertible (i.e., $\det(G^TG) \neq 0$) the system admits a unique least square solution, i.e., a unique solution of $G^TG\mathbf{x} = G^T\mathbf{b}$. The corresponding portfolio is called the **least square hedging portfolio** and it is the portfolio that, in the least square sense, better approximates a self-financing hedging portfolio. Note that the least square hedging portfolio is in general neither hedging nor self-financing! However it is, in the least square sense, the "best" approximation of an hedging, self-financing portfolio. For example if we let $S_0 = B_0 = 100$, $e^u = 1.05$, $e^r = 1.03$ and we consider a call with strike $K = S_0 = 100$, then the possible pay-offs at time 2 are

$$
\begin{aligned}
&Y(u,u) = 10.25, \quad Y(u,m) = Y(m,u) = 5,\\
&Y(u,d) = Y(m,m) = Y(d,u) = Y(m,d) = Y(d,m) = Y(d,d) = 0
\end{aligned}
$$

and the least-square hedging portfolio is

$$
\begin{aligned}
&h_S(2,u) = 0.9903,\ h_S(2,m) = 0.5371,\ h_S(2,d) = -0.0112\\
&h_B(2,u) = -0.9335,\ h_B(2,m) = -0.4899,\ h_B(2,d) = 0.0095,\\
&h_S(1) = 0.8119,\ h_B(1) = -0.7533.
\end{aligned}
$$

Thus if the stock price goes up at time 1 and time 2, the value of the portfolio at maturity is

$$h_S(2,u)S_0e^{2u} + h_B(2,u)B_0e^{2r} \approx 10.14$$

which is actually not enough to hedge the derivative, since $Y(u,u) = 10.25$. Hence the seller must add the cash 0.11 at maturity to pay-off the buyer. Moreover, still along the path (u,u), there is a cash flow at time 1 given by

$$C(1) = h_S(1)S_0e^u + h_B(1)B_0e^r - (h_S(2,u)S_0e^u + h_B(2,u)B_0e^r) \approx -0.17$$

which is negative, meaning that the seller has added this amount to the portfolio. Hence along this path the seller would incur in the loss $0.17 + 0.11 = 0.28$. This of course is the worst case scenario for the writer as the path (u,u) gives the maximum value of the pay-off.

We also remark that the initial value of the portfolio is

$$V(0) = h_S(1)S_0 + h_B(1)B_0 \approx 5.86$$

and this could be interpreted as the "fair price" at time 0 of the derivative, according to this "almost self-financing hedging" portfolio.

Chapter 3

A project on the Asian option

The risk-neutral pricing formula for European call and put options, and for other simple standard European derivatives, reduces to a simple expression involving the standard normal distribution, see (43) for the case of European calls. For Asian options, and other path-dependent options, this reduction is not possible, and the application of numerical methods to valuate these derivatives becomes essential. The Monte Carlo numerical method is the most popular among practitioners. This project deals with applications of the Monte Carlo method to compute the risk-neutral value of Asian options.

The Asian option

The Asian call/put option in the time-continuum case is defined as the non-standard European derivative with pay-off

$$Y^{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+, \qquad Y^{\text{put}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+,$$

where $K > 0$ is the strike price of the option. The Black-Scholes price at time $t = 0$ of these options is given by

$$\Pi_{\text{AC}}(0) = e^{-rT}\mathbb{E}_q[Y^{\text{call}}], \quad \Pi_{\text{AP}}(0) = e^{-rT}\mathbb{E}_q[Y^{\text{put}}].$$

Task 3.1 (*)**.** *Derive the following put-call parity identity:*

$$\Pi_{\text{AC}}(0) - \Pi_{\text{AP}}(0) = e^{-rT}\left(\frac{e^{rT}-1}{rT}S_0 - K\right). \tag{3.1}$$

Task 3.2 (*)**.** *The Asian call with geometric average is the non-standard European derivative with pay-off*

$$Q = \left(e^{\frac{1}{T}\int_0^T \log S(t)\,dt} - K\right)_+. \tag{3.2}$$

Show that the Black-Scholes price at time $t = 0$ of this derivative is given by

$$\Pi_{\mathrm{AC}}^{(\mathrm{G})}(0) = e^{-rT}(e^{qT} S_0 \Phi(d_1) - K\Phi(d_2)) \tag{3.3a}$$

where

$$q = \frac{1}{2}(r - \frac{\sigma^2}{6}), \quad d_2 = d_1 - \sigma\sqrt{\frac{T}{3}}, \quad d_1 = \frac{\log \frac{S_0}{K} + \frac{1}{2}(r + \frac{\sigma^2}{6})T}{\sigma\sqrt{T/3}}.$$

HINT: You need to apply Theorem 0.10.

Monte Carlo valuation of the Asian option

Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$. We approximate the pay-off of the Asian option on the given partition as

$$Y = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+ \approx \left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K\right)_+ .$$

Task 3.3 (Matlab)**.** *Write a Matlab code which computes the Black-Scholes price at time $t = 0$ and the confidence interval of the Asian option using the crude Monte Carlo method. Write also a code which applies the control variate Monte Carlo method using the pay-off of the Asian option with geometric mean as control variate. Compare the new method with crude Monte Carlo method and show that the control variate technique improves the performance of the computation. Finally use the control variate Monte Carlo method to study numerically how the price of the Asian call depend on the parameters of the option. In particular:*

(a) Verify numerically the put-call parity (3.1)

(b) Show that the Asian call is less sensitive to volatility than the standard call. Do you have an intuitive explanation for this?

(c) Show that for large volatility the Monte Carlo method becomes unstable (the confidence interval grows very fast)

(d) Show that the Asian call is cheaper than the standard call with the same strike. Do you have an intuitive explanation for this?

Chapter 4

A project on coupon bonds

Coupon bonds are debt instruments issued by national governments as a way to borrow money and fund their activities. Given the long maturity of coupon bonds (which can reach up to 30 or more years), the valuation of these contracts must take into account the time fluctuation of the risk-free rate. Once a stochastic model for the risk-free rate is prescribed, the valuation of coupon bonds can be carried out using the so called "classical approach", which is based on the risk-neutral pricing formula. The main purpose of this project is to numerically compute the yield curve of coupon bonds implied by a particular example of stochastic risk-free rate model.

4.1 Zero-coupon bonds

A **zero-coupon bond** (ZCB) with **face** (or **nominal**) value K and maturity $T > 0$ is a contract that promises to pay to its owner the amount K at time T in the future. Zero-coupon bonds, and the related coupon bonds described in Section 4.2, are issued by national governments and private companies as a way to borrow money and fund their activities. Without loss of generality we assume from now on that $K = 1$, as owning a ZCB with face value K is clearly equivalent to own K shares of a ZCB with face value 1. Moreover in the following we assume that all ZCB's are issued by one given institution, so that all bonds differ merely by their maturities.

After being originally issued in the so-called **primary market**, the ZCB becomes a tradable asset in the **secondary** bond market. It is therefore natural to model the value at time t of the ZCB maturing at time $T > t$ as a random variable, which we denote by $B(t,T)$. Hence $\{B(t,T)\}_{t\in[0,T]}$ is a stochastic process. We assume throughout the discussion that the institution issuing the bond bears no risk of default, i.e., $B(t,T) > 0$, for all $t \in [0,T]$. Clearly $B(T,T) = 1$ and, under normal market conditions, $B(t,T) < 1$, for $t < T$, i.e., ZCB's are risk-free assets ensuring a positive return. However exceptions are possible; for instance national bonds in Sweden with maturity shorter than 5 years yield currently (2017) a negative return. A **ZCB market** is a market that consists of ZCB's with different maturities. Our main goal is to introduce models for the price of ZCB's observed in the market. For modeling purposes we assume that there is a ZCB in the market maturing at each time $T \in [0, S]$, where S is the maturity of the latest expiring ZCB in the market (e.g., $S \approx 30$ years). Note that this assumption is quite far from reality, one reason being that bonds with maturity larger than, say, 2 years will most likely pay coupons.

Interest rates and yield of ZCB's

The difference in value of ZCB's with different maturities is expressed through the implied forward rate of the bond. To define this concept, suppose that at the present time t we open a portfolio that consists of -1 share of a ZCB with maturity $t < T$ and $B(t,T)/B(t,T+\delta)$ shares of a ZCB expiring at time $T+\delta$. This investment has zero value and entails that we pay 1 at time T and receive $B(t,T)/B(t,T+\delta)$ at time $T+\delta$. Hence our investment at the present time t is equivalent to an investment in the future time interval $[T, T+\delta]$ with (annualized) return given by

$$F_\delta(t,T) = \frac{1}{\delta}\left(B(t,T)/B(t,T+\delta) - 1\right) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}. \tag{4.1}$$

The quantity $F_\delta(t,T)$ is also called **discretely compounded forward rate** in the interval $[T, T+\delta]$ **locked at time** t (or forward LIBOR, as it is commonly applied to LIBOR interest rate contracts). The name is intended to emphasize that the investment return in the future interval $[T, T+\delta]$ is locked at the *present* time $t \leq T$, that is to say, we know today which interest rate has to be charged to borrow in the future time interval $[T, T+\delta]$ (if a different rate were locked today, then an arbitrage opportunity would arise). When $\delta \to 0^+$ we obtain the **continuously compounded T-forward rate**

$$f(t,T) = \lim_{\delta \to 0^+} \frac{1}{\delta} \frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)} = -\partial_T \log B(t,T), \tag{4.2}$$

which is the rate locked at time t to borrow at time T for an "infinitesimal" period of time. In the following we shall consider only continuously compounded rates.

The curve $T \to f(t,T)$ is called **forward rate curve** of the ZCB market. The knowledge of the forward rate curve determines the price $B(t,T)$ of all ZCB's in the market through the formula

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\, ds\right), \quad 0 \leq t \leq T \leq S, \tag{4.3}$$

which follows easily by integrating (4.2).

The quantity

$$r(t) = f(t,t), \quad t \in [0,S] \tag{4.4}$$

is called the **(continuously compounded) spot rate** of the ZCB market at time t and represents the interest rate locked at time t to borrow instantaneously at time t (i.e., on the spot).

The spot rate can be used to define the **discount process**:

$$d(t) = \exp\left(-\int_0^t r(s)\,ds\right) \quad \text{(continuously compounded)} \tag{4.5}$$

If t is the present time and $X(\tau)$ is the value of an asset at some given future time $\tau > t$, then the quantity

$$\frac{d(\tau)}{d(t)}X(\tau) = \exp\left(-\int_t^\tau r(s)\,ds\right)X(\tau)$$

is called the present (at time t) **discounted value** of the asset and represents the future (at time τ) value of the asset relative to the purchasing value of money at that time.

We conclude this section by presenting the fundamental concept of ZCB's yield to maturity. The (continuously compounded) **yield (to maturity)** $y(t,T)$ at time t of the ZCB with maturity T is the *constant* forward rate which entails the value $B(t,T)$ of the ZCB. Hence the yield $y(t,T)$ of a ZCB is obtained by replacing $f(t,v) = y(t,T)$ in (4.3), i.e.,

$$B(t,T) = e^{-y(t,T)(T-t)}, \quad \text{i.e.,} \quad y(t,T) = -\frac{\log B(t,T)}{T-t} \tag{4.6}$$

To put it in other words: Selling a ZCB for the price $B(t,T)$ at time t (i.e., borrowing $B(t,T)$ at time t) is equivalent to lock the constant forward rate $y(t,T)$ until maturity. Note also that the first equation in (4.6) expresses $B(t,T)$ as the discounted value at time t of the future payment $= 1$ at maturity assuming that the spot rate is constant and equal to $y(t,T)$ in the interval $[t,T]$.

4.2 Coupon bonds

Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0,T]$. A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \ldots, c_M \in [0,1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1+c_M$ at maturity $T = t_M$. Note that some c_k may be zero, which means that no coupon is actually paid at that time. We set $c = (c_1, \ldots, c_M)$ and denote by $B_c(t,T)$ the value at time t of the bond paying the coupons $c_1, \ldots, c_M$ and maturing at time T. Now, let $t \in [0,T]$ and $k(t) \in \{1, \ldots, M\}$ be the smallest index such that $t_{k(t)} > t$, that is to say, $t_{k(t)}$ is the first time after t at which a coupon is paid. Holding the coupon bond at time t is clearly equivalent to holding a portfolio containing $c_{k(t)}$ shares of the ZCB expiring at time $t_{k(t)}$, $c_{k(t)+1}$ shares of the ZCB expiring at time $t_{k(t)+1}$, and so on, hence

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j B(t,t_j) + (1 + c_M)B(t,T), \tag{4.7}$$

the sum being zero when $k(t) = M$.

The yield of a coupon bond is defined implicitly by the equation

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j e^{-y_c(t,T)(t_j-t)} + (1+c_M)e^{-y_c(t,T)(T-t)} \tag{4.8}$$

and so *the yield of the coupon bond is the constant spot rate used to discount the total future payments of the coupon bond.*

Remark 4.1. Most commonly the coupons are equal. Letting $c_j = c$, for all $j = 1, \dots, M$, the formula (4.8) simplifies to

$$B_c(t,T) = c \sum_{j=k(t)}^{M-1} e^{-y_c(t,T)(t_j-t)} + (1+c)e^{-y_c(t,T)(T-t)}. \tag{4.9}$$

To compute the yield of a coupon bond with values $B_c(t,T)$, one has to invert (4.9). For instance, assume that $T = M$ years and that the coupons are paid annually, that is $t_1 = 1$, $t_2 = 2, \dots, t_M = M$. Then $x = e^{-y_c(0,T)}$ solves $p(x) = 0$, where p is the M-order polynomial given by

$$p(x) = c_1 x + c_2 x^2 + \dots + (1+c_M)x^M - B_c(0,T). \tag{4.10}$$

The roots of this polynomial can easily be computed numerically, e.g., with the command `roots[p]` in matlab, see Task 4.4 below.

Yield curve

(Zero) coupon bonds are listed in the market in terms of their yield rather than in terms of their price. The curve $T \to y_c(t,T)$ is called the **yield curve** of the ZCB market. Figure 4.1 shows an example of yield curve for governmental Swedish bonds.

Task 4.1 (*). *Yield curves observed in the market are classified based on their shape (e.g., steep, flat, inverted, etc.). Find out on the Internet what the different shapes mean from an economical point of view and write a short text (one page) about this.*

4.3 The classical approach to ZCB's pricing

In this section we describe the so-called **classical approach** to ZCB's pricing. This approach is based on the risk-neutral pricing formula.

Definition 4.1. *Let $\{r(t)\}_{t\geq 0}$ be a stochastic process modeling the spot interest rate of the ZCB market, where we assume that $r(0) = r_0$ is a deterministic constant. Then*

$$B(0,T) = \mathbb{E}[d(T)] = \mathbb{E}[e^{-\int_0^T r(s)\,ds}]. \tag{4.11}$$

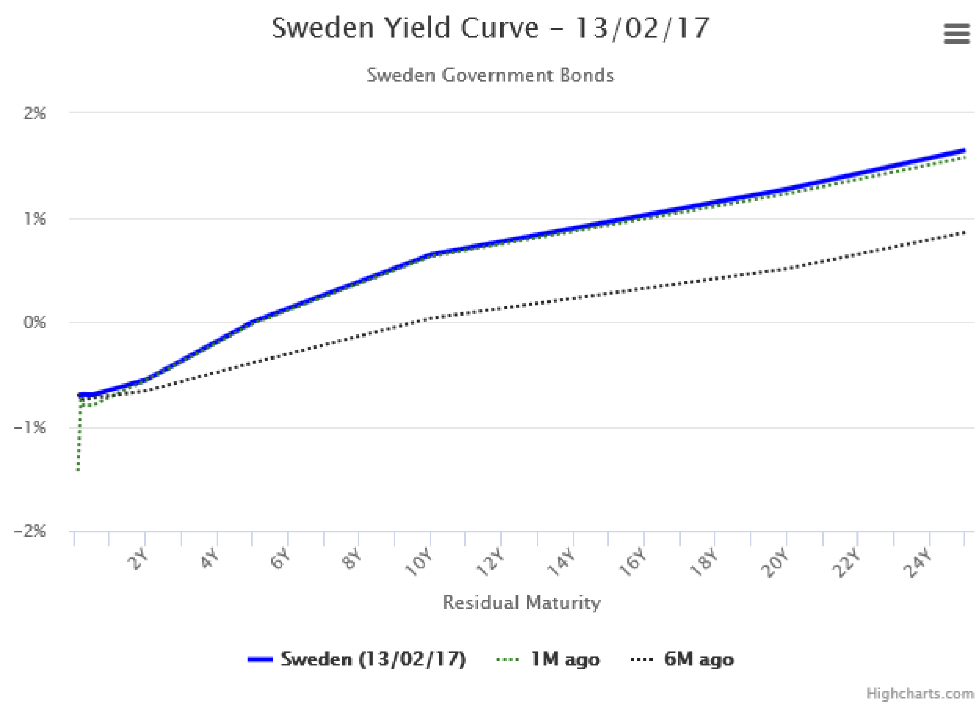


Figure 4.1: Yield curve for Swedish bonds. Note that the yield is negative for maturities shorter than 5 years. Bonds with maturity larger than 2 years have coupon and thus their yield is computed using (4.8) (instead of (4.6)).

Hence the value at time $t = 0$ of the ZCB is the expected value of the discounted future payment $= 1$ of the ZCB. Note that in a purely ZCB market, one cannot define a martingale, or risk-neutral, probability, hence the expectation in (4.11) is taken in the physical probability. Equivalently, in the classical approach to ZCB's pricing, the physical probability is assumed to be risk-neutral.

In the following two tasks it is asked to compute the exact value of $B(0, T)$ when the spot rate is given by two simple models, namely the Ho-Lee model and the Vasicek model.

Task 4.2 (*). *In the* **Ho-Lee model**, *the risk-free rate is assumed to satisfy*

$$r(t) = r(0) + \theta(t) + \sigma W(t), \quad \theta(0) = 0, \tag{4.12}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion, $\sigma > 0$ is constant and $\theta(t)$ is a deterministic function of time. Derive the initial price $B(0, T)$ of the ZCB with face value 1 and maturity $T > 0$ and the forward rate $f(0, T)$ implied by the Ho-Lee model. Sketch the graph $T \to f(0, T)$ of the forward curve at time $t = 0$ (experiment for different functions θ). HINT: You need Theorem 0.10.

Task 4.3 (*)**.** *In the* **Vasicek model**, *the risk-free rate is assumed to satisfy*

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma W(t) - a\sigma \int_0^t e^{a(s-t)} W(s)\, ds, \tag{4.13}$$

where $\{W(t)\}_{t\geq 0}$ *is a Brownian motion and* $a > 0, b \in \mathbb{R}, \sigma > 0$ *are constants. Note that, by Theorem 0.10,* $r(t)$ *is normally distributed. Show that the initial price of the ZCB with face value 1 and maturity* $T > 0$ *is given by*

$$B(0,T) = e^{-r(0)A(T)+C(T)}, \tag{4.14a}$$

where

$$A(T) = \frac{1}{a}(1 - e^{-aT}), \tag{4.14b}$$

$$C(T) = \left(b - \frac{\sigma^2}{2a^2}\right)(A(T) - T) - \frac{\sigma^2}{4a}A(T)^2. \tag{4.14c}$$

HINT: You need Theorem 0.10.

Remark 4.2. By using the notation in Remark 0.9, we can write the definition of $r(t)$ in the Vasicek model as

$$r(t) = r(0) + b(e^{at} - 1) + \sigma \int_0^t e^{as} dW(s),$$

which is the form of $r(t)$ most used in the literature.

Task 4.4 (Matlab)**. Part I.** *Write a matlab function*

```
yield(B, Coupon, FirstCoupDate, CoupFreq, T)
```

that computes the continuously compounded yield of a coupon bond. Here, `B` *is the current (i.e., at time* $t = 0$*) price of the coupon bound,* `Coupon` $\in [0, 1)$ *is the (constant) coupon,* `FirstCoupDate` *is the first future date at which the coupon is paid,* `CoupFreq` *is the frequency of coupon payments and* `T` *is the time left to maturity maturity. For example*[1]

$$\texttt{yield}(1.01, 0.02, 46/252, 1, 19 + 201/252)$$

computes the yield of a 2% coupon bond which today is valued 1.01, pays the first coupon in 46 days and matures in 19 years and 201/252 days. **Part II.** *Let* $\{r(t)\}_{t\geq 0}$ *be given by the Vasicek model with parameters* a, b, σ*. Apply the code in Part I to perform a parameter sensitivity analysis of the yield curve. Can you reproduce all the typical shapes found in Exercise 4.1?*

[1]Remember that time in finance is measured in fraction of years and 1 year = 252 days (unless otherwise stated in the contract).

Chapter 5

A project on multi-asset options

Multi-asset options are financial derivatives on several underlying assets. In this project we consider standard European derivatives on two stocks, i.e., European style derivatives with pay-off of the form $Y = g(S_1(T), S_2(T))$, where $S_1(t), S_2(t)$ are the prices of the underlying stocks at time $t \in [0, T]$ and $T > 0$ is the time of maturity of the derivative. After the Black-Scholes theory for single stock options is generalized to the two dimensional case, the Monte Carlo method is applied to compute the price of maximum call options on two stocks.

5.1 Examples of options on two stocks

Given $K_1, K_2 > 0$, a **two assets correlation** call option with maturity T is the European derivative with pay-off

$$Y = \begin{cases} (S_2(T) - K_2)_+ & \text{if } S_1(T) > K_1 \\ 0 & \text{otherwise} \end{cases} .$$

A **maximum** call option on two stocks with maturity T is the European style derivative with pay-off $Y = \max((S_1(T) - K_1)_+, (S_2(T) - K_2)_+)$, and similarly one defines the minimum call option on two stocks and the analogous put options.

The European derivative with maturity T and pay-off

$$Y = (S_1(T) - S_2(T))_+$$

is called a **spread** option (or **exchange asset** option). The European derivative with maturity T and pay-off

$$Y = \left(\frac{S_1(T)}{S_2(T)} - K \right)_+$$

is called a **relative outperformance** option.

A **quanto** option is a call or put option on a stock in which the pay-off is paid in a different currency than the one in which the stock is traded. Thus, letting $\Xi(t)$ be the exchange rate of the two currencies at time t, the pay-off of a quanto call option with maturity T is

$$Y = \Xi(T)(S(T) - K)_+.$$

Note that in this example the second asset is not at stock but a market index (the exchange rate $\Xi(t)$).

The list of examples could go on, but we stop here. New types of multi-asset options are created frequently. All multi-asset options are traded OTC.

5.2 Black-Scholes price of 2-assets standard European derivatives

In Black-Scholes theory of two-dimensional markets it is assumed that the stocks prices are given by a 2-dimensional geometric Brownian motion, namely:

$$S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_{11} W_1(t) + \sigma_{12} W_2(t)}, \tag{5.1a}$$

$$S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_{21} W_1(t) + \sigma_{22} W_2(t)}, \tag{5.1b}$$

where $\{W_1(t)\}_{t\geq 0}, \{W_2(t)\}_{t\geq 0}$ are independent Brownian motions in the physical probability $\mathbb{P}$ and $\alpha_1, \alpha_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ are real constants. We assume that the **volatility matrix** $\sigma = (\sigma_{ij})$ is invertible. Letting

$$W(t) = (W_1(t), W_2(t)), \quad \sigma_1 = (\sigma_{11}, \sigma_{12}), \quad \sigma_2 = (\sigma_{21}, \sigma_{22}),$$

we can rewrite the 2-dimensional geometric Brownian motion in the more concise form

$$S_j(t) = S_j(0)e^{\alpha_j t + \sigma_j \cdot W(t)},$$

where $\cdot$ denotes the standard scalar product of vectors. We start by deriving the joint density of the stocks prices.

Theorem 5.1. *The random variables* $\log S_1(t), \log S_2(t)$ *are jointly normally distributed with mean* $m = (\log S_1(0) + \alpha_1 t, \log S_2(0) + \alpha_2 t)$ *and covariant matrix* $C = t\sigma\sigma^T$. *In particular, the random variables* $S_1(t), S_2(t)$ *have the joint density*

$$f_{S_1(t),S_2(t)}(x, y) = \frac{e^{-\frac{1}{2t}\left(\log\frac{x}{S(0)} - \alpha_1 t \quad \log\frac{y}{S(0)} - \alpha_2 t\right)(\sigma\sigma^T)^{-1}\begin{pmatrix}\log\frac{x}{S(0)} - \alpha_1 t \\ \log\frac{y}{S(0)} - \alpha_2 t\end{pmatrix}}}{txy\sqrt{(2\pi)^2 \det(\sigma\cdot\sigma^T)}}. \tag{5.2}$$

Proof. We have

$$\log S_1(t) = \log S_1(0) + \alpha_1 t + \sigma_{11} W_1(t) + \sigma_{12} W_2(t),$$
$$\log S_2(t) = \log S_2(0) + \alpha_2 t + \sigma_{21} W_1(t) + \sigma_{22} W_2(t),$$

hence the first statement of the theorem follows by Theorem 0.9. The joint density of $S_1(t), S_2(t)$ is computed using that

$$\begin{aligned} F_{S_1(t),S_2(t)}(x,y) &= \mathbb{P}(S_1(t) \le x, S_2(t) \le y) \\ &= \mathbb{P}(\log S_1(t) \le \log x, \log S_2(t) \le \log y) = F_{\log S_1(t), \log S_2(t)}(\log x, \log y), \end{aligned}$$

hence

$$\begin{aligned} f_{S_1(t),S_2(t)}(x,y) &= \partial_x \partial_y F_{S_1(t),S_2(t)}(x,y) = \partial_x \partial_y [F_{\log S_1(t),\log S_2(t)}(\log x, \log y)] \\ &= (xy)^{-1} f_{\log S_1(t),\log S_2(t)}(\log x, \log y), \end{aligned}$$

which, using the joint normal density of $\log S_1(t)$ and $\log S_2(t)$, gives (5.2). □

The geometric Brownian motion is often given in a different but equivalent (in distribution) form, as shown in the next Task.

Task 5.1 (*). *Show that the process* (5.1) *is equivalent, in distribution, to the process*

$$S_1(t) = S_1(0) e^{\alpha_1 t + |\sigma_1| W_1(t)} \tag{5.3a}$$

$$S_2(t) = S_2(0) e^{\alpha_2 t + |\sigma_2| (\rho W_1(t) + \sqrt{1-\rho^2} W_2(t))}, \tag{5.3b}$$

where $|\sigma_i| = \sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}$ *is the Euclidean norm of the vector* σ_i *and*

$$\rho = \frac{\sigma_1 \cdot \sigma_2}{|\sigma_1||\sigma_2|} \in [-1, 1] \tag{5.4}$$

is the cosine of the angle between σ_1, σ_2.

Remark 5.1. It can be shown the historical variances of the two stocks are unbiased estimators for $|\sigma_1|^2, |\sigma_2|^2$, while the historical correlation of log-returns of the two stocks is an unbiased estimator for ρ.

Now, exactly as in the one-dimensional case discussed in Section 0.4, it is possible to define a risk-neutral (or martingale) probability measure with respect to which the discounted value of both stocks are martingales. We seek this probability measure in the class of Girsanov probabilities $\mathbb{P}_\theta$ introduced in Theorem 0.14, where $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. The problem is solved in the following two theorems, which are the 2-dimensional generalizations of Theorem 0.17 and Theorem 0.18 respectively.

Theorem 5.2. *Let $\mu = (\alpha_1 - r + \frac{1}{2}|\sigma_1|^2, \alpha_2 - r + \frac{1}{2}|\sigma_1|^2)$. The discounted values $S_1^*(t) = e^{-rt}S_1(t)$ and $S_2^*(t) = e^{-rt}S_2(t)$ of the stocks have constant expectation in the Girsanov probability $\mathbb{P}_\theta$ if and only if $\theta = q = (q_1, q_2)$, where q is the (unique) solution of the linear system $\sigma q = \mu$.*

Theorem 5.3. *The stochastic processes $\{S_1^*(t)\}_{t\geq 0}$ and $\{S_2^*(t)\}_{t\geq 0}$ are martingales in the probability measure $\mathbb{P}_\theta$ if and only if $\theta = q = (q_1, q_2)$.*

Task 5.2 (*). *Prove Theorem 5.2.*

The probability measure $\mathbb{P}_q$ is the martingale (or risk-neutral) probability of the 2-dimensional Black-Scholes market.

Task 5.3 (*). *Prove that in the probability $\mathbb{P}_q$ the stocks prices are given by the following 2-dimensional geometric Brownian motion*

$$S_1(t) = S_1(0)e^{(r-\frac{|\sigma_1|^2}{2})t+\sigma_1\cdot W^{(q)}(t)}, \tag{5.5a}$$

$$S_2(t) = S_2(0)e^{(r-\frac{|\sigma_2|^2}{2})t+\sigma_2\cdot W^{(q)}(t)}, \tag{5.5b}$$

where $W^{(q)}(t) = (W_1^{(q)}(t), W_2^{(q)}(t)) = (W_1(t)+q_1t, W_2(t)+q_2t)$ and $\{W_1^{(q)}(t)\}_{t\geq 0}, \{W_1^{(q)}(t)\}_{t\geq 0}$ are $\mathbb{P}_q$-independent Brownian motions.

Denoting by $\mathbb{E}_q$ the expectation in the probability $\mathbb{P}_q$, the Black-Scholes price at time $t = 0$ of the 2-assets European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]$.

Example: relative outperformance options

For the relative outperformance call option with strike K the pay-off function is given by

$$g(x, y) = \left(\frac{x}{y} - K\right)_+.$$

Let's compute the Black-Scholes price at time $t = 0$ using the risk-neutral pricing formula $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[g(S_1(T), S_2(T))]$, i.e., using (5.5),

$$\begin{aligned}\Pi_Y(0) &= e^{-rT}\mathbb{E}_q\left[\left(\frac{S_1(T)}{S_2(T)} - K\right)_+\right] \\ &= e^{-rT}\mathbb{E}_q\left[\left(\frac{S_1(0)}{S_2(0)}e^{(\frac{|\sigma_2|^2}{2}-\frac{|\sigma_1|^2}{2})T+(\sigma_1-\sigma_2)\cdot W^{(q)}(T)} - K\right)_+\right],\end{aligned}$$

where we recall that $W^{(q)}(t) = (W_1^{(q)}(t), W_2^{(q)}(t))$ and $\{W_1^{(q)}(t)\}_{t\geq 0}$, $\{W_1^{(q)}(t)\}_{t\geq 0}$ are independent Brownian motion in the risk-neutral probability measure. Now we write

$$(\sigma_1 - \sigma_2) \cdot W^{(q)}(T) = \sqrt{T}[(\sigma_{11} - \sigma_{21})G_1 + (\sigma_{12} - \sigma_{22})G_2] = \sqrt{T}(X_1 + X_2),$$

where $G_j = W_j^{(q)}(T)/\sqrt{T} \in \mathcal{N}(0,1)$, $j = 1,2$, hence $X_j \in \mathcal{N}(0, (\sigma_{1j} - \sigma_{2j})^2)$, $j = 1,2$. In addition, X_1, X_2 are independent random variables, hence $X_1 + X_2$ is normally distributed with zero mean and variance $(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 = |\sigma_1 - \sigma_2|^2$ (see Theorem 0.9). It follows that

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q\left[\left(\frac{S_1(0)}{S_2(0)}e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})T + \sqrt{T}|\sigma_1 - \sigma_2|G)} - K\right)_+\right],$$

where $G \in \mathcal{N}(0,1)$. Hence, letting

$$\hat{r} = \frac{|\sigma_1 - \sigma_2|^2}{2} + \left(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2}\right)$$

and $a = e^{(\hat{r}-r)\tau}$, we have

$$\Pi_Y(0) = ae^{-\hat{r}T}\mathbb{E}_q\left[\left(\frac{S_1(0)}{S_2(0)}e^{(\hat{r} - \frac{|\sigma_1 - \sigma_2|^2}{2})T + \sqrt{T}|\sigma_1 - \sigma_2|G} - K\right)_+\right].$$

Up to the multiplicative parameter a, this is the Black-Scholes price at time $t = 0$ of a call on a stock with price $S_1(0)/S_2(0)$, volatility $|\sigma_1 - \sigma_2|$ and for an interest rate of the money market given by $\hat{r}$. Hence

$$\Pi_Y(0) = a\left(\frac{S_1(0)}{S_2(0)}\Phi(d_+) - Ke^{-\hat{r}T}\Phi(d_-)\right) := v_0(S_1(0), S_2(0)), \tag{5.6}$$

where

$$d_\pm = \frac{\log\frac{S_1(t)}{KS_2(t)} + (\hat{r} \pm \frac{|\sigma_1 - \sigma_2|^2}{2})\tau}{|\sigma_1 - \sigma_2|\sqrt{\tau}}.$$

Task 5.4 (Matlab)**.** *Write a Matlab function*

$$\texttt{MaximumCall}(\texttt{K}_1, \texttt{K}_2, \texttt{T}, \texttt{s1}, \texttt{s2}, \texttt{sigma1}, \texttt{sigma2}, \texttt{rho}, \texttt{r}, \texttt{N}, \texttt{n})$$

which applies the crude Monte Carlo method to compute the initial price (at time $t = 0$) of the maximum call option with strikes K_1, K_2 and expiring at time T. Here `s1`, `s2` *are the initial prices of the stocks,* `sigma1`, `sigma2` *are the volatilities of the stocks and* `rho` *is their correlation (in the notation* (5.3), `sigma1`$= |\sigma_1|$*). Moreover N is the number of points in a uniform partition of* $[0,T]$ *and* n *is the number of paths used for the Monte Carlo simulation. Use this function to perform a parameter sensitivity analysis of the maximum call option.*

Integral form of the Black-Scholes price for general standard European derivatives on two stocks

In the case of standard European derivatives the risk-neutral pricing formula $\Pi_Y(0) = e^{-rT}\mathbb{E}[g(S_1(T), S_2(T)]$ can be written in a closed integral form, as shown in the following analog of Theorem 0.19.

Theorem 5.4. *The Black-Scholes price at time $t = 0$ of the 2-stocks option with pay-off $Y = g(S_1(T), S_2(T))$ is given by*

$$\Pi_Y(0) = v_0(S_1(0), S_2(0)), \tag{5.7a}$$

where the pricing function v_0 is given by

$$v_0(x,y) = \int_{\mathbb{R}^2} g\left(xe^{(r-\frac{|\sigma_1|^2}{2})T+\sqrt{T}\xi}, ye^{(r-\frac{|\sigma_2|^2}{2})T+\sqrt{T}\eta}\right) \frac{\exp\left(-\frac{1}{2}\begin{pmatrix}\xi & \eta\end{pmatrix}(\sigma\sigma^T)^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)}{2\pi\sqrt{\det(\sigma\sigma^T)}} d\xi\, d\eta. \tag{5.7b}$$

Sketch of the proof. The proof follows by using the joint density of $S_1(T), S_2(T)$ in the risk-neutral probability to compute $\mathbb{E}_q[Y]$. Namely, according to (5.5), the stock prices in the risk-neutral probability are given by a geometric Brownian motion with mean of log-returns $\alpha_j = r - \frac{1}{2}|\sigma_j^2|$, $j = 1, 2$. Replacing these values of α_1, α_2 into (5.2) gives the joint density $\widetilde{f}_{S_1(t),S_2(t)}(x,y)$ of the stock prices in the risk-neutral probability, from which we can compute $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[g(S_1(T), S_2(T)]$ as

$$\Pi_Y(0) = e^{-rT} \int_{\mathbb{R}^2} g(x,y) \widetilde{f}_{S_1(T),S_2(T)}(x,y)\, dx\, dy.$$

After a proper change of variable, the previous expression transforms into (5.7). □

The Black-Scholes price at time $t > 0$ of the 2-assets European derivative with pay-off $Y = g(S_1(T), S_2(T))$ is given by a formula similar to the one in Theorem 5.4, namely

$$\Pi_Y(t) = v(t, S_1(t), S_2(t)),$$

where the pricing function v is given by

$$v(t,x,y) = \int_{\mathbb{R}^2} g\left(xe^{(r-\frac{|\sigma_1|^2}{2})\tau+\sqrt{\tau}\xi}, ye^{(r-\frac{|\sigma_2|^2}{2})\tau+\sqrt{\tau}\eta}\right) \frac{\exp\left(-\frac{1}{2}\begin{pmatrix}\xi & \eta\end{pmatrix}(\sigma\sigma^T)^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)}{2\pi\sqrt{\det(\sigma\sigma^T)}} d\xi\, d\eta. \tag{5.8}$$

Moreover it can be shown that the number of shares on the two stocks in the self-financing hedging portfolio of the derivative is given by

$$h_{S_1}(t) = \partial_x v(t, S_1(t), S_2(t)), \quad h_{S_2}(t) = \partial_y v(t, S_1(t), S_2(t)),$$

while the number of shares $h_B(t)$ on the risk-free asset is determined by the replicating condition $\Pi_Y(t) = h_{S_1}(t)S_1(t) + h_{S_2}(t)S_2(t) + h_B(t)B(t)$, i.e.,

$$h_B(t) = B(t)^{-1}(\Pi_Y(t) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t)).$$

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