MVE171 Basic Stochastic Processes and Financial Applications

Written exam Saturday 7 December 2019 8.30 AM–11.30 AM

TEACHER: Patrik Albin. JOUR: MV PhD-student, telephone 031 7725325.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. **(a)** Calculate $\mathbf{P}\{X(1) = Y(2)\}$ when $X(t)$ and $Y(t)$, $t \in \mathbb{R}$, are independent zero-mean WSS Gaussian processes with common autocorrelation functions $R_X(\tau) = R_Y(\tau) = \mathrm{e}^{-|\tau|}$. **(2.5 points)**

(b) Let $X(t)$ and $Y(t)$, $t \geq 0$, be independent Poission processes with intensity (/rate) 1. Find $\mathbf{P}\{X(1) = Y(2)\}$. **(2.5 points)**

Task 2. Let $X(n)$, $n = 0, 1, 2, \ldots$, be a Markov chain with possible values (/states) 0 and 1, transition matrix $P = \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$ and starting distribution $p(0) = (1\ \ 0)$. Find the mean $\mathbf{E}\{T\}$ of the random time $T = \min\{n > 0 : X(m) = 1 \text{ for some } 0 < m < n \text{ and } X(n) = 0\}$ it takes for the chain to make a journey to 1 and back to 0. [HINT: A discrete random variable ξ with PMF $P_\xi(k) = \mathbf{P}\{\xi = k\} = p\,(1-p)^{k-1}$ for $k = 1, 2, \ldots$ has mean $\mathbf{E}\{\xi\} = 1/p$.] **(5 points)**

Task 3. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent identically distributed discrete random variables with possible values $\{-1, 1\}$ and $\mathbf{P}\{X_k = 1\} = 1 - \mathbf{P}\{X_k = -1\} = p \in [0, 1]$. Show that M_n, $n \geq 0$, given by $M_0 = 0$ and $M_n = \sum_{k=1}^{n} X_k - n\,(2p-1)$ for $n \geq 1$ is a martingale with respect to the filtration $F_n = \sigma(X_0, \ldots, X_n)$. **(5 points)**

Task 4. Consider an M/M/1/3 queueing system with one server and two queueing slots with $\lambda = \mu = \rho = 1$. Write a computer programme that by means of stochastic simulation finds an approximation of the probability that the server is busy during ten straight time units (when the queueing system is in steady state as usual). [HINT: $p_0 = p_1 = p_2 = p_3 = 1/4$.] **(5 points)**

MVE171 Solutions to written exam 7 December 2019

Task 1. **(a)** We have $\mathbf{P}\{X(1)=Y(2)\}=\mathbf{P}\{X(1)-Y(2)=0\}=\mathbf{P}\{\mathrm{N}(\mu,\sigma^2)=0\}=0$ as $\mu=\mathbf{E}\{X(1)\}-\mathbf{E}\{Y(2)\}=0$ and $\sigma^2=\mathbf{E}\{(X(1)-Y(2))^2\}=\mathbf{E}\{X(1)^2)\}+\mathbf{E}\{Y(2)^2\}$ $=2\,\mathrm{e}^{-|0|}=2>0$.

(b) $\mathbf{P}\{X(1)=Y(2)\}=\sum_{k=0}^{\infty}\mathbf{P}\{X(1)=k\}\,\mathbf{P}\{Y(2)=k\}=\sum_{k=0}^{\infty}\frac{1^k}{k!}\,\mathrm{e}^{-1}\,\frac{2^k}{k!}\,\mathrm{e}^{-2}=\sum_{k=0}^{\infty}\frac{2^k}{(k!)\cdot(k!)}\,\mathrm{e}^{-3}$.

Task 2. $\mathbf{E}\{T\}=1/(1/2)+1/(1/2)=4$.

Task 3. We have $\mathbf{E}\{M_{n+1}|F_n\}=\mathbf{E}\{X_{n+1}\}+\sum_{k=1}^{n}X_k-(n+1)\,(2p-1)=M_n$ since $\mathbf{E}\{X_{n+1}\}=2p-1$.

Task 4.

```
In[1]:= Reps=1000000; Busy=0;

In[2]:= For[i=1, i≤Reps, i++, X=RandomInteger[{0,3}];
            If[X≤2, time=Random[ExponentialDistribution[2]],
              If[X==3, time=Random[ExponentialDistribution[1]]]];
    While[(X≥1) && (time<10),
      If[X≤2, time=time+Random[ExponentialDistribution[2]];
          X=X+2*(RandomInteger[{0,1}]-1/2),
        time=time+Random[ExponentialDistribution[1]]; X=2]];
    If[X≥1, Busy=Busy+1]];

In[3]:= N[Busy/Reps]

Out[3]:= 0.099792
```