MVE171 Basic Stochastic Processes and Financial Applications

Written exam Monday 6 April 2020 8.30 AM–11.30 AM

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Aids: All aids are permitted. (See the Canvas announcement with istructions for Eastern reexam for clarification.)

Grades: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Task 1. Consider an M/M/1/3 queueing system with one server and two queueing slots with $\lambda = 1$ and $\mu = 1/2$. Write a computer programme that by means of stochastic simulation finds an approximation of the mean of the time until the system becomes empty (when the queueing system is in steady state as usual). **(5 points)**

Task 2. Consider a Poisson process $X(t)$, $t \geq 0$. Is it possible to have $\mathbf{P}\{X(1) = 0, X(2) = 0, \ldots, X(n) = 0\} = \mathbf{P}\{X(1) = 2, X(2) = 4, \ldots, X(n) = 2n\}$? **(5 points)**

Task 3. Let X_n, $n = 0, 1, 2, \ldots$, be a discrete time Markov chain with transition probability matrix P. Is it true in general that $(P^{-1})_{ij} = \mathbf{P}\{X_n = i | X_{\,n+1} = j\}$? Can the same claim be true for some special case of P? **(5 points)**

Task 4. Let $W(t)$, $t \geq 0$, be a Wiener process that is independent of a unit intensity (/rate) Poisson process $N(t)$, $t \geq 0$. Show that $M(t) = \mathrm{e}^{2t}(-1)^{N(t)}W(t)$, $t \geq 0$, is a martingale with respect to the filtration F_t containing information of all process values $\{W(s)\}_{s\in[0,t]}$ and $\{N(s)\}_{s\in[0,t]}$. **(5 points)**

MVE171 Solutions to written exam 6 April 2020

Task 1.

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In[1]:= reps=100000; totaltime=0;

In[2]:= For[i=1, i≤reps, i++, time=0; x=RandomReal[];

    If[x≤1/15, X=0, If[x≤3/15, X=1, If[x≤7/15, X=2, X=3]]];

    While[X≥1,

      If[X≤2, atime=Random[ExponentialDistribution[1]];

          stime=Random[ExponentialDistribution[1/2]];

          time=time+Min[atime,stime];

          If[atime≤stime, X=X+1, X=X-1],

        time=time+Random[ExponentialDistribution[1/2]]; X=2]];

    totaltime=totaltime+time];

In[3]:= totaltime/reps

Out[3]:= 18.9295
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Task 2. With λ the intensity of the process the asked for equality means that $\mathbf{P}\{X(1) = 0\}\,\mathbf{P}\{X(2)-X(1) = 0\}\cdot\ldots\cdot\mathbf{P}\{X(n)-X(n-1) = 0\} = \mathbf{P}\{X(1) = 2\}\,\mathbf{P}\{X(2)-X(1) = 2\}\cdot\ldots\cdot\mathbf{P}\{X(n)-X(n-1) = 2\}$ which holds if and only if $\mathbf{P}\{X(1) = 0\} = \mathrm{e}^{-\lambda}$ equals $\mathbf{P}\{X(1) = 2\} = \frac{\lambda^2}{2!}\,\mathrm{e}^{-\lambda}$ so that it is necessary and sufficient that $\lambda = \sqrt{2}$.

Task 3. As $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ the claim is not true in general. However, the claim is trivially true for P the identity matrix.

Task 4. $\mathbf{E}\{\mathrm{e}^{2t}(-1)^{N(t)}W(t)|F_s\} = \mathrm{e}^{2t}(-1)^{N(s)}\,\mathbf{E}\{(-1)^{N(t)-N(s)}(W(t)-W(s))\} + \mathrm{e}^{2t} \times (-1)^{N(s)}W(s)\,\mathbf{E}\{(-1)^{N(t)-N(s)}\} = 0 + \mathrm{e}^{2t}(-1)^{N(s)}W(s)\sum_{k=0}^{\infty}(-1)^k\,\frac{(t-s)^k}{k!}\,\mathrm{e}^{-(t-s)} = \mathrm{e}^{2s}(-1)^{N(s)}W(s)$ for $0 \leq s \leq t$.