MVE172 Basic Stochastic Processes and Financial Applications

Written exam Saturday 5 December 2020 8.30–11.30 AM

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AIDS: All aids are permitted. (See the Canvas course "Ordinarie tentamen Modul: 0220, MVE172" with instructions for this reexam for clarifications.)

GRADES: 8 points for grade 3, 12 points for grade 4 and 16 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Is the process $Y(t) = \mathrm{e}^{X(t)}$ WSS when $\{X(t)\}_{t\in\mathbb{R}}$ is a WSS Gaussian process? [HINT: Stationary implies WSS for processes in general and WSS implies stationary for Gaussian processes.] **(5 points)**

Task 2. Consider a Markov chain $\{X_k\}_{k=0}^{\infty}$ with possible values $\{0,1,2\}$, initial state probabilities $p(0) = (1/3\ \ 1/3\ \ 1/3)$ and all transition probabilitites $p_{ij} = 1/3$. Calculate $\mathbf{E}\{T\}$ for $T = \min\{k \geq 0 : X_k = 2\}$. **(5 points)**

Task 3. Calculate the probability $\mathbf{P}\{X_1(1/2)X_2(1)X_3(3/2) = 2\}$ when $X_1(t)$, $X_2(t)$ and $X_3(t)$, $t \geq 0$, are independent $\mathbb{N}$-valued continuous time random processes with $\mathbf{P}\{X_1(t) = k\} = \mathbf{P}\{X_2(t) = k\} = \mathbf{P}\{X_3(t) = k\} = \frac{t^k}{k!}\,\mathrm{e}^{-t}$ for $k = 0, 1, 2, \ldots$.

(5 points)

Task 4. Let $\{X(t)\}_{t\geq 0}$ be a zero-mean Gaussian stationary independent increment process with autocorrelation function $R_X(s,t) = \min(s,t)$. What functions $f(t) > 0$ make $\{f(t)\cos(X(t))\}_{t\geq 0}$ a martingale with respect to the filtration $F_t = \sigma(X(s) : s \in [0,t])$. [HINT: $\mathbf{E}\{\cos(\mathrm{N}(0,t))\} = \mathrm{e}^{-t/2}$.]

(5 points)

MVE172 Solutions to written exam December 2020

Task 1. As $X(t)$ is WSS Gaussian it is stationary and then also $\mathrm{e}^{X(t)}$ is stationary and therefore WSS.

Task 2. With probability $1/3$ we have $X_0 = 2$ so that $T = 0$. Otherwise, on the average it takes the chain the mean 3 of a waiting time distribution with $p = 1/3$ to move from the states $\{0, 1\}$ to the state 2. Hence $\mathbf{E}\{T\} = 1/3 \cdot 0 + 2/3 \cdot 3 = 2$.

Task 3. The possible values of $(X_1(1/2), X_2(1), X_3(3/2))$ are $(2, 1, 1)$, $(1, 2, 1)$ and $(1, 1, 2)$ with probability $\big(\frac{(1/2)^2}{2!} \cdot \frac{1^1}{1!} \cdot \frac{(3/2)^1}{1!} + \frac{(1/2)^1}{1!} \cdot \frac{1^2}{2!} \cdot \frac{(3/2)^1}{1!} + \frac{(1/2)^1}{1!} \cdot \frac{1^1}{1!} \cdot \frac{(3/2)^2}{2!}\big)\, \mathrm{e}^{-1/2-1-3/2}$ $= \frac{9}{8}\, \mathrm{e}^{-3}$.

Task 4. As $\mathbf{E}\{\cos(X(t))|F_s\} = \mathbf{E}\{\cos(X(t)-X(s)) \cos(X(s))|F_s\} - \mathbf{E}\{\sin(X(t)-X(s)) \times \sin(X(s))|F_s\} = \cos(X(s))\,\mathbf{E}\{\cos(X(t) - X(s))\} - \sin(X(s))\,\mathbf{E}\{\sin(X(t) - X(s))\} = \cos(X(s))\,\mathrm{e}^{-(t-s)/2} - \sin(X(s)) \cdot 0$ for $0 \leq s \leq t$ it is required that $f(t) = C\,\mathrm{e}^{t/2}$ for some constant $C > 0$ [as we required $f(t)$ to be positive].