MVE172 Basic Stochastic Processes and Financial Applications

Written home re-exam Wednesday 7 April 2021 8.30–11.30 AM

TEACHER: Patrik Albin. JOUR: Petar Jovanovski petarj@chalmers.se 072 9499722.

AIDS: All aids are permitted. (See the Canvas course "Omtentamen 1 Modul: 0220, MVE172" with instructions for this reexam for clarifications.)

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(t)\}_{t\geq 0}$ be a Poisson process with rate 1, $\{Y(t)\}_{t\geq 0}$ a zero-mean WSS Gaussian process with autocorrelation function $R_Y(\tau) = \mathrm{e}^{-|\tau|}$ that is independent of the X-process and S and T independent unit mean exponential distributed random times that are independent of the X- and Y-processes. Find an expression for $\mathbf{P}\{X(S)Y(T) > x\}$ for $x > 0$. [HINT: Recall that $\int_0^\infty z^k \,\mathrm{e}^{-z}\, dz = k!$ for $k \in \mathbb{N}$.] **(5 points)**

Task 2. Let $N(t)$ be the number of customers at time $t \geq 0$ in an M/M/1/3 queueing system with $\lambda = \mu = 1$ such that $N(0) = 0$. Find by means of stochastic simulation an approximation of the expected value of the time $T = \min\{t > 0 : N(t) = 0,\ N(s) = 3 \text{ for some } s \in (0,t)\}$ it takes to move from empty queueing system to full queueing system and back to empty queueing system again. (Analytic solutions give zero points.) **(5 points)**

Task 3. Kal and Ada start with initial fortune 2 trillion SEK each and repeatedly play a game where Karl wins with probability $p \in (0,1)$ and Ada wins with probability $1-p$ and where the winner gets 1 trillion SEK from the looser after each game. The gaming goes on until one of Kal and Ada has no money left and the other one of them is declared the winner. Find the probability that Kal wins. **(5 points)**

Task 4. Which discrete time martingales $\{M_n\}_{n=0}^\infty$ with $\mathbf{E}\{M_n^2\} < \infty$ for all n are also WSS processes (so that $\mathbf{E}\{M_m\}$ and $\mathbf{E}\{M_m M_{m+n}\}$ do not depend on m)? [HINT: Show that $\mathbf{E}\{M_n^2\} = \mathbf{E}\{(M_n - M_m)^2\} + \mathbf{E}\{M_m^2\}$ for $0 \leq m \leq n$.] **(5 points)**

MVE172 Solutions to written re-exam 7 April 2021

Task 1. $\mathbf{P}\{X(S)Y(T)>x\} = \sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\mathbf{P}\{N(0,k^2)>x\}\,\mathrm{e}^{-s}\mathrm{e}^{-t}\,\mathbf{P}\{X(s)=k\}\,dsdt = \sum_{k=1}^{\infty}\int_0^{\infty}(1-\Phi(x/k))\,\mathrm{e}^{-s}\,\frac{s^k}{k!}\,\mathrm{e}^{-s}\,ds = \sum_{k=1}^{\infty}2^{-(k+1)}\,(1-\Phi(x/k))$.

Task 2. It is not very hard to prove analytically that the sought after expectation is 12. Turning to stochastic simulations we use that the sought after expectation is two times the expectation $\mathbf{E}\{\min\{t>0: N(t)=3\}\}$ which we simulate as

```
In[1]:= Reps = 1000000;
In[2]:= For[i=1; time=0, i<=Reps, i++, Nt=0;
          While[Nt<3,
            If[Nt==0, Nt=1;
               time=time+Random[ExponentialDistribution[1]],
            If[Nt==1, If[Random[]<1/2, Nt=0, Nt=2];
               time=time+Random[ExponentialDistribution[2]],
            If[Nt==2, If[Random[]<1/2, Nt=1, Nt=3];
               time=time+Random[ExponentialDistribution[2]]]]]]];
In[3]:= 2*time/Reps
Out[3]= 11.9954
```

Task 3. Let P_i be the probability that Kal wins when he starts gaming with i trillion SEK for $i=1,2,3$. Then we have

$$P_1 = p\cdot P_2, \qquad P_2 = p\cdot P_3 + (1-p)\cdot P_1 \qquad \text{and} \qquad P_3 = p + (1-p)\cdot P_2.$$

This system of equations we solve to get $P_2 = p^2/(1-2p+2p^2)$.

Task 4. As $\mathbf{E}\{M_n^2\}-\mathbf{E}\{(M_n-M_m)^2\}-\mathbf{E}\{M_m^2\} = 2\,\mathbf{E}\{(M_n-M_m)M_m\} = 2\,\mathbf{E}\{\mathbf{E}\{(M_n-M_m)M_m|F_m\}\} = 2\,\mathbf{E}\{M_m\mathbf{E}\{M_n-M_m|F_m\}\} = 0$ for $0\le m\le n$ we must have $\mathbf{E}\{(M_n-M_m)^2\} = 0$ to make $\mathbf{E}\{M_m^2\}$ not depend on m implying that $M_n = M_0$ for all n. On the other hand $M_n = M_0$ is both a martingale and WSS so that is the answer.