MVE172 Basic Stochastic Processes and Financial Applications

Written exam Saturday 4 December 2021 8.30–11.30 AM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let Y_1, Y_2, Y_3 and Y_4 be independent standard normal random variables (with zero expectation and unit variance) and put $X_1 = (Y_1 + Y_2)/\sqrt{2}$, $X_2 = (Y_1 + Y_2 + Y_3 + Y_4)/2$ and $X_3 = (Y_1 - Y_2)/\sqrt{2}$. State with reason if the following is true:

(a) X_1, X_2 and X_3 are all standard normal random variables. **(1,25 points)**

(b) X_i, $i = 1, 2, 3$, is a Gaussian random process. **(1,25 points)**

(c) X_1 and X_2 are independent random variables. **(1,25 points)**

(d) X_1 and X_3 are independent random variables. **(1,25 points)**

Task 2. Let $\{X_k\}_{k=0}^{\infty}$ be a (time homogeneous) Markov chain. Is it true that $\mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}, \ldots, X_{k+n} = i_{k+n}\} = \mathbf{P}\{X_k = i_k | X_{k+1} = i_{k+1}\}$ for $n \geq 1$? **(5 points)**

Task 3. Let $\{X_i\}_{i=1}^{\infty}$ be random variables and $F_n = \sigma(X_1, \ldots, X_n)$. Under what conditions is $M_n = \sum_{i=1}^{n} X_i$ a martingale with respect to F_n? **(5 points)**

Task 4. Let T_1 and T_2 be the times for the first and second jump of a Poisson process with rate $\lambda > 0$. Find the joint probability density function of T_1 and T_2. HINT: $\mathbf{P}\{T_1 \leq s, T_2 \leq t\} = \mathbf{P}\{X(s) \geq 1,\ X(t) \geq 2\}$. **(5 points)**

MVE172 Solutions to written exam 4 December 2021

Task 1. **(a)** As all X_i's are linear combination of independent normal random variables they are normal distributed. Calculations of means and variances give zero and one.

(b) All linear combination of the X_i's are linear combination of the independent normal distributed Y_i's and therefore are normal distributed. Hence X_i, $i=1,2,3$, is Gaussian.

(c) X_1 and X_2 are not independent because $\mathbf{Cov}\{X_1,X_2\}=1/\sqrt{2}\neq 0$.

(d) X_1 and X_3 are independent because they are jointly normal with $\mathbf{Cov}\{X_1,X_3\}=0$.

Task 2. $\mathbf{P}\{X_k=i_k|X_{k+1}=i_{k+1},\ldots,X_{k+n}=i_{k+n}\}=\frac{\mathbf{P}\{X_k=i_k\}\,p_{i_k,i_{k+1}}p_{i_{k+1},i_{k+2}}\cdots p_{i_{k+n-1},i_{k+n}}}{\mathbf{P}\{X_{k+1}=i_{k+1}\}\,p_{i_{k+1},i_{k+2}}\cdots p_{i_{k+n-1},i_{k+n}}}$ $=\frac{\mathbf{P}\{X_k=i_k\}\,p_{i_k,i_{k+1}}}{\mathbf{P}\{X_{k+1}=i_{k+1}\}}=\mathbf{P}\{X_k=i_k|X_{k+1}=i_{k+1}\}$ so yes.

Task 3. $\mathbf{E}\{|X_n|\}<\infty$ and $\mathbf{E}\{X_{n+1}|F_n\}=0$ for all n.

Task 4. $f_{T_1,T_2}(s,t)=\frac{\partial^2}{\partial s\partial t}\mathbf{P}\{X(s)\geq 1,\,X(t)\geq 2\}=\frac{\partial^2}{\partial s\partial t}\mathbf{P}\{X(s)=1\}\,\mathbf{P}\{X(t-s)\geq 1\}+\frac{\partial^2}{\partial s\partial t}\mathbf{P}\{X(s)\geq 2\}=\ldots=\lambda^2\mathrm{e}^{-\lambda t}$ for $0\leq s\leq t$.