1. See lecture 10, slide 16.
2. See lecture 8, slide 8.
3. (a) $\pi(\theta|y) \propto \pi(y|\theta)\pi(\theta) \propto \theta^y e^{-\theta}\theta^{\alpha-1}e^{-\beta\theta} = \theta^{y+\alpha-1}e^{-(\beta+1)\theta} \propto \Gamma(\theta, y+\alpha-1, \beta+1)$.
   (b) $\pi(\theta|y) \propto \pi(y|\theta)\pi(\theta) \propto (1-\theta)^r\theta^y\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{\alpha+y-1}(1-\theta)^{\beta+r-1} \propto \mathcal{B}(\theta, \alpha+y, \beta+r)$
   (c) The posterior $\pi(\theta|y_1, \mu_\theta, \sigma_\theta) = \mathcal{N}(\theta, \hat{\mu}, \hat{\sigma}^2)$ where

$$\hat{\sigma}^2 = \left(\frac{1}{\sigma_\theta^2} + \frac{1}{\sigma^2}\right)^{-1},$$
$$\hat{\mu} = \hat{\sigma}^2\left(\frac{\mu_\theta}{\sigma_\theta^2} + \frac{y_1}{\sigma^2}\right)$$

4. (a) $h(X_{0:n}) = \mathbb{I}(X_{0:n} \in A)$ since $\mathbb{E}_{0:n}[\mathbb{I}(X_{0:n} \in A)] = \mathbb{P}(X_{0:n} \in A)$.
   (b) See Lecture 7, slide 25.
5.

$$\pi(\lambda_i) = \Gamma(\lambda_i, 1, 0.1)\,, i = 1, 2,$$
$$\pi(p) \propto 1.$$
$$f(\mathbf{y}|\lambda_1, \lambda_2, p) = \prod_{i=1}^{n}\left((1-p)\Gamma(y_i;\alpha,\beta_0) + (1-p)\Gamma(y_i;\alpha,\beta_1)\right)$$

(a) A suitable auxiliary variables is $z_i \sim Bin(1, p)$ Since then the joint posterior is

$$\pi(\mathbf{z}, \mathbf{y}, \beta_0, \beta_1, p) = \Gamma(\beta_0, 1, 0.1)\Gamma(\beta_1, 1, 0.1)\prod_{i=1}^{n}(1-p)^{z_i}p^{1-z_i}, \Gamma(y_i;\beta_{z_i}).$$

which is possible to sample from.

b) Since $\mathbb{P}(z_i = 0) \propto p\Gamma(y_i;\beta_0)$ and $\mathbb{P}(z_i = 1) \propto (1-p)\Gamma(y_i;\alpha,\beta_1)$:

$$z_i|\ldots \sim Bin\left(1, \frac{p\Gamma(y_i|\beta_0)}{p\Gamma(y_i;\beta_0) + (1-p)\Gamma(y_i;\beta_1)}\right)$$

For β_1

$$\pi(\mathbf{z}, \mathbf{y}, \beta_0, \beta_1, p) \propto e^{-0.1\beta_1}\prod_{i=1}^{N}\beta_1^{z_i\alpha}e^{-\beta_1 z_i y_i} = \beta_1^{\alpha\sum_{i=1}^n z_i}e^{-(\sum_{i=1}^n z_i y_i + 0.1)\beta_1}$$

Thus

$$\lambda_1|\ldots \sim \Gamma(1\alpha\sum_{i=1}^n z_i, \sum_{i=1}^n z_i y_i + 0.1).$$

The distribution of λ_2 follows by same reasoning.
Finally, for p $\pi(\mathbf{z}, \mathbf{y}, \lambda_1, \lambda_2, p) \propto (1-p)^{\sum z_i}(p)^{n-\sum z_i}$ thus

$$p|\ldots \sim Beta(n - \sum z_i + 1, \sum z_i + 1).$$

c) Let $\boldsymbol{\theta} = \{\beta_0, \beta_1, p\}$ The posterior distribution equals:

$$\pi(\boldsymbol{\theta}|\mathbf{y}) \propto \Gamma(\beta_0, 1, 0.1)\Gamma(\beta_1, 1, 0.1)\prod_{i=1}^{n}\left((1-p)\Gamma(y_i;\alpha,\beta_0) + p\Gamma(y_i;\alpha,\beta_1)\right).$$

And since the accpetance rate equals:

$$\begin{aligned}\alpha(\boldsymbol{\theta}^*, \boldsymbol{\theta}) &= \min\left(1, \frac{\pi(\boldsymbol{\theta}^*|\mathbf{y})q(\boldsymbol{\theta}|\boldsymbol{\theta}^*)}{\pi(\boldsymbol{\theta}|\mathbf{y})q(\boldsymbol{\theta}^*|\boldsymbol{\theta})}\right) = \min\left(1, \frac{\pi(\boldsymbol{\theta}^*|\mathbf{y})}{\pi(\boldsymbol{\theta}|\mathbf{y})}\right) \\ &= \frac{\Gamma(\beta_0^*, 1, 0.1)\Gamma(\beta_1^*, 1, 0.1)\prod_{i=1}^n \left((1-p^*)\Gamma(y_i;\alpha,\beta_0^*) + p^*\Gamma(y_i;\alpha,\beta_1^*)\right)}{\Gamma(\beta_0, 1, 0.1)\Gamma(\beta_1, 1, 0.1)\prod_{i=1}^n \left((1-p)\Gamma(y_i;\alpha,\beta_0) + p\Gamma(y_i;\alpha,\beta_1)\right)}\end{aligned}$$

where the last equality is since MHRW is symmetric.

6. The variance of the IS estimator, τ, is

$$\mathbb{E}_g\left[x^2\frac{f^2(x)}{g^2(x)}\right] - \tau^2,$$

and since

$$\mathbb{E}_g\left[x^2\frac{f^2(x)}{g^2(x)}\right] \propto \int_c^{\infty} x^2 e^{-x^2+\frac{x^2}{2\sigma^2}}\,dx$$

which is unbounded if $\sigma \leq \frac{1}{2}$. Thus one can only use the proposal distribution if $\sigma > \frac{1}{2}$.