Matematiska vetenskaper
Chalmers tekniska högskola
.

Exam: 2015-10-24, Time: 14:00
Computer Intensive Statistical methods
(MSA100/MVE186)

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Allowed aids: None.

Correct, and **well motivated solutions** gives points indicated within the parentheses at each exercise. Total number of points is 49. The grade limits are 3:19, 4:28 and 5:38 points, respectively and G:19, and VG:35.

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1. Show that for the bivariate distribution $\pi(x_1, x_2)$, the two stage Gibbs sampler has π as stationary distribution. (5p)

2. Show that Multinomial resampling strategy adds no bias to self normalizing importance sampling. (5p)

3. (a) Assume that $y \sim \pi(y|\theta) = Po(\theta)$, show that $\Gamma(\alpha, \beta)$ is a conjugate prior for θ. (3p)

   (b) Assume that $y \sim \pi(y|\theta) = NB(r, \theta)$, where r is known. Show that $Beta(\alpha, \beta)$ is a conjugate prior for θ. (3p)

   (c) Let $y \sim \mathcal{N}(\theta, \sigma^2)$, show that $\pi(\theta) \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$ is a conjugate prior to θ. Derive the parameters for the posterior distribution given that one observes y_1. (4p)

4. In this exercise we will be studying a Markov chain $\{X_i\}_{i=0}^N$ with transition kernel $q(x|y)$ and initial distribution π_0

   (a) We are interested in estimating $\mathbb{P}(X_{0:n} \in A)$, using a SISR method. What function, $h(X_{0:n})$ should we use to estimate the probability in SISR algorithm? (3p)

   (b) Write down the pseudo algorithm of the SISR sampler. (Either general form or special case when the density of interest is a Markov chain) (6p)

5. We are interested ins sampling from the posterior distribution of the following model:

$$\pi(\beta_k) = \Gamma(\beta_k, 1, 0.1)\,, k = 0, 1,$$
$$\pi(p) \propto 1,$$
$$\pi(\mathbf{y}|\beta_0, \beta_1, p, \alpha) = \prod_{i=1}^{n} \left((1-p)\Gamma(y_i; \alpha, \beta_0) + p\Gamma(y_i; \alpha, \beta_1)\right),$$

   where α is assumed to be known constant The goal of this exercise is to generate the posterior distribution using MCMC.

   (a) Suggest auxiliary variables $\mathbf{z}$ which makes it possible to Gibbs sample the posterior. Derive the joint distribution $\pi(\mathbf{z}, \mathbf{y}, \beta_0, \beta_1, p)$. (4p)

   (b) Derive the conditional distribution for each variable in its parametric form. (4p)

   (c) Instead of Gibbs sampling one can use Metropolis Hasting random walk. Derive the acceptance probability as a function β_0, β_1, p. (4p)

6. Assume that one want to estimate $\mathbb{E}_f[x]$, where $f(x) \propto e^{-\frac{x^2}{2}}\mathbb{I}(x > c), c > 0$, using IS. For proposal distribution it suggested using $g(x) \propto e^{-\frac{x^2}{2\sigma^2}}$. Can one use this proposal for any σ? (8p)

- Normal distribution.

Notation	$\mathcal{N}(x;\mu,\sigma^2)$
Suport	$x \in \mathbb{R}$
parameter	$\sigma^2 > 0, \mu \in \mathbb{R}$
pdf	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

- Beta distribution.

Notation	$\mathcal{B}(x;\alpha,\beta)$
Suport	$x \in [0,1]$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

- Gamma distribution.

Notation	$\Gamma(x;\alpha,\beta)$
Suport	$x \in [0,\infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

- inverse Gamma distribution.

Notation	$IG(x;\alpha,\beta)$
Suport	$x \in [0,\infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta x^{-1}}$

- Exponential distribution.

Notation	$exp(x;\lambda)$
Suport	$x \in [0,\infty)$
parameter	$\lambda > 0$
pdf	$\lambda e^{-\lambda x}$
cdf	$1 - e^{-\lambda x}$

- Raised cosine

Notation	$Rcos(x;\mu)$
Suport	$x \in [\mu - 1, \mu + 1]$
parameter	$\mu \in \mathbb{R}$
pdf	$\frac{1}{2}[1 + \cos(\pi(x-\mu))]$

- Poisson

Notation	$Pois(x; \lambda)$
Suport	$x \in \mathbb{N}$
parameter	$\lambda > 0$
pdf	$\frac{\lambda^x}{x!} e^{-\lambda}$

- Binomial

Notation	$Bin(x; n, p)$
Suport	$x \in \{0, 1, \dots, n\}$
parameter	$p \in [0, 1], n \in \mathbb{N}$
pdf	$\binom{n}{x}(1-p)^{n-x} p^x$

- Negative binomial

Notation	$NB(x; r, p)$
Suport	$x \in \{0, 1, \dots\}$
parameter	$p \in [0, 1], r \in \mathbb{N}$
pdf	$\binom{x+r-1}{x}(1-p)^r p^x$