Matematiska vetenskaper **Fake**
Chalmers tekniska högskola **MVE186**

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Tillåtna hjälpmedel: Nej

Correct, well motivated solution gives points indicated within the parenthesis at each exercise. Total number of points is 42. The grade limits are 3:17, 4:27 and 5:34 points.

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1. Show that MH algorithm satisfies the global balance condition. (5p)

2. Define and prove the Law of large number for independent identically distributed random variables. (5p)

3. Using only standard normal ($\mathcal{N}(0,1)$) and uniform ($U[0,1]$) random variables. Write an algorithm that generates a random variable from the following distributions

   (a) Weibull distribution, (2p)

   (b) $\mathcal{B}(2,2)$, (2p)

   (d) $U[a,b]$. (2p)

4. We have the following model

$$\begin{aligned}
\pi(\alpha,\beta) &\propto 1,\\
\pi(p_i) &= \mathcal{B}(p;\alpha,\beta),\\
f(y_i|p_i) &= \begin{cases} (1-p_i)^{y_i-1}p_i & \text{if } 0 < y_i < 10 \\ \sum_{k=10}^{\infty}(1-p_i)^{k-1}p_i & \text{if } y_i = 10. \end{cases}, \; i = 1,\ldots,n
\end{aligned}$$

   That is the likelihood comes from a censored geometric distribution. The goal of this exercise is to generate samples from the posterior distribution of the parameters using MCMC.

   (a) Suggest an auxiliary variable $\mathbf{z}$ making it possible to use a Gibbs sampler for sampling the posterior distribution. Derive the joint distribution $f(\mathbf{z},\mathbf{y},\mathbf{p},\alpha,\beta)$. (4p)

   b) Derive each conditional distribution. (4p)

   c) To improve convergence it turns out that it is better to sample α,β jointly, this can not be done using a Gibbs sampler, instead a MH random walk is suggested. Derive the acceptance probability as a function of $\alpha,\beta,\mathbf{p}$. (4p)

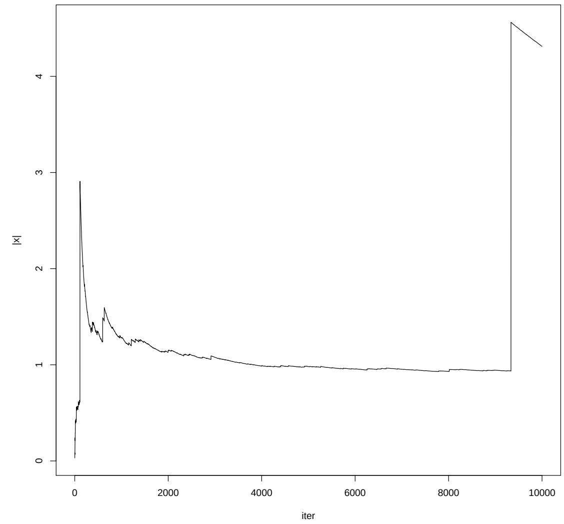

Figure 1: Importance sampling trajectory

5. (a) Figure 1 below shows the convergence of $\tau_N = \sum_{i=1}^{N} h(X_i)\omega(X_i; f, g)$ for a standard importance sampler. Here f, g are known densities, h a known function and $\omega(X_i; f, g) = \frac{f(X_i)}{g(X_i)}$. Give a plausible reason for why the trajectory of τ_N has a jump. (2p)

   (b) Figure 2 below shows a trace plot of Markov Chain generated by MH random walk. What is causing this problem? Suggest an improvement. (2p)

   (c) Explain what Effective sample size is. (2p)

6. In the algorithm below, f, g are normalized densities with $f(x) \leq Kg(x)$.

   repeat
   draw $X^* \sim g$
   draw $U \sim U[0, 1]$
   until $U \leq \frac{f(X^*)}{Kg(X^*)}$
   $X \leftarrow X^*$
   return X

   (a) Derive the parametric distribution of number samples from g required in the algorithm. (4p)

   (b) Express the parameter of the distribution in a) as a function of f, g, K. (4p)

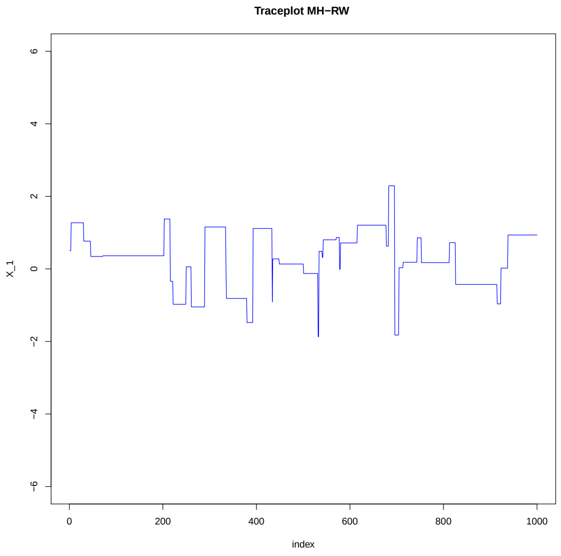

Figure 2: Importance sampling trajectory

- Normal distribution.

Notation	$\mathcal{N}(x;\mu,\sigma^2)$
Suport	$x \in \mathbb{R}$
parameter	$\sigma^2 > 0, \mu \in \mathbb{R}$
pdf	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

- Beta distribution.

Notation	$\mathcal{B}(x;\alpha,\beta)$
Suport	$x \in [0,1]$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

- Gamma distribution.

Notation	$\Gamma(x;\alpha,\beta)$
Suport	$x \in [0,\infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

- inverse Gamma distribution.

Notation	$IG(x;\alpha,\beta)$
Suport	$x \in [0,\infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta x^{-1}}$

- Exponential distribution.

Notation	$exp(x;\lambda)$
Suport	$x \in [0, \infty)$
parameter	$\lambda > 0$
pdf	$\lambda e^{-\lambda x}$
cdf	$1 - e^{-\lambda x}$

- Weibull distribution.

Notation	$W(x;\lambda,k)$
Suport	$x \in [0, \infty)$
parameter	$\lambda > 0, k > 0$
pdf	$\frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}e^{-(\frac{x}{\lambda})^k}$
cdf	$1 - e^{-(\frac{x}{\lambda})^k}$

- Raised cosine

Notation	$Rcos(x;\mu)$
Suport	$x \in [0, \infty)$
parameter	$\mu \in \mathbb{R}$
pdf	$\frac{1}{2}[1 + \cos(\pi(x - \mu))]$

- Poisson

Notation	$Pois(x;\lambda)$
Suport	$x \in \mathbb{N}$
parameter	$\lambda > 0$
pdf	$\frac{\lambda^x}{x!}e^{-\lambda}$

- Geometric

Notation	$Geo(x;p)$
Suport	$x \in \mathbb{N} - \{0\}$
parameter	$p \in [0,1]$
pdf	$(1-p)^{x-1}p$

- Binomial

Notation	$Bin(x;n,p)$
Suport	$x \in \{0,1,\ldots,n\}$
parameter	$p \in [0,1], n \in \mathbb{N}$
pdf	$\binom{n}{x}(1-p)^{n-x}p^x$

- Special

Notation	$f_s(x;\beta,\alpha,n)$
Suport	$x > 0$
parameter	$\beta > 0, \alpha > 0, n \in \mathbb{N}$
pdf	$\left(\frac{\Gamma(x+\beta)}{\Gamma(x)}\right)^n e^{-\alpha x}$