Matematiska vetenskaper **Exam: 2015-01-05, Time: 14:00**
Chalmers tekniska högskola **Computer Intensive Statistical methods**
. **(MSA100/MVE186)**

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Allowed aids: None.

Correct, well motivated solution gives points indicated within the parentheses at each exercise. Total number of points is 42. The grade limits are 3:18, 4:24 och 5:32 points.

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1. State and prove the Law of large numbers for Markov chains.

   One can assume that Chebyshev's inequality. Which states that for a non negative random variable Y with $\mathbb{E}[Y^2] < \infty$ the following is true

   $$\mathbb{P}(|Y - \mathbb{E}[Y]| \geq \epsilon) \leq \frac{\mathbb{V}[Y]}{\epsilon^2},$$

   for any $\epsilon > 0$. (5p)

2. Show that the Metropolis-Hastings algorithm satisfies the global balance condition. (5p)

3. Using only standard normal ($\mathcal{N}(0,1)$) and uniform ($U[0,1]$) random variables, write down algorithms that draws random numbers from:

   (a) Rayleigh distribution. (2p)

   (b) density with pdf $f(x) \propto x\mathcal{I}(x \in [0,2])$. (2p)

   (c) Truncated Exponential, $f(x) \propto \exp(x;1)\mathcal{I}(x \in [1,\infty))$. (2p)

4. What is the definition of a control variate (cv)? For a random variable to be used as a cv, which properties of the random variable needs to be known in advance and which properties needs to at least be estimated? (6p)

5. Bernoulli regression: We have the following Bayesian model

   $$\pi(\boldsymbol{\beta}) = \mathcal{N}(\boldsymbol{\beta}; \mathbf{0}, \mathbf{I}),$$
   $$f(y_i|\boldsymbol{\beta}, \sigma^2) = Bin(y_i; 1, p_i),\ i = 1, \dots, n,$$

   where $p_i = \Phi(\mathbf{X}_i\boldsymbol{\beta})$ and where $\mathbf{X}_i$ is the ith row of $\mathbf{X}$. The goal of this exercise is to generate the posterior distribution of $\boldsymbol{\beta}$ using MCMC.

   (a) Suggest auxiliary variables $\mathbf{z}$ which makes it easy to Gibbs sample $\boldsymbol{\beta}$. Derive the joint distribution $f(\mathbf{z}, \mathbf{y}, \boldsymbol{\beta})$. (4p)

   b) Derive the conditional distributions $\mathbf{z}|\boldsymbol{\beta}$ and $\boldsymbol{\beta}|\mathbf{z}$. (4p)

   c) An alternative is to sample everything directly using the Metropolis-Hastings algorithm. Derive the acceptance probability for $\boldsymbol{\beta}$ if one is using a MH random walk. (4p)

6. Suppose we have the following time series:

$$x_t = \mu + \rho x_{t-1} + \sqrt{V_t} Z_t,$$

where $\rho = -0.5$, V_t is $\Gamma(1,1)$ distributed and Z_t a standard normal random variable. What is the effective sample size of $\frac{1}{T}\sum_{t=1}^{T} x_t$? (8p)

- Normal distribution.

Notation	$\mathcal{N}(x;\mu,\sigma^2)$
Suport	$x \in \mathbb{R}$
parameter	$\sigma^2 > 0, \mu \in \mathbb{R}$
pdf	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

- Beta distribution.

Notation	$\mathcal{B}(x;\alpha,\beta)$
Suport	$x \in [0,1]$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

- Gamma distribution.

Notation	$\Gamma(x;\alpha,\beta)$
Suport	$x \in [0,\infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

- inverse Gamma distribution.

Notation	$IG(x; \alpha, \beta)$
Suport	$x \in [0, \infty)$
parameter	$\alpha > 0, \beta > 0$
pdf	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta x^{-1}}$

- Exponential distribution.

Notation	$exp(x; \lambda)$
Suport	$x \in [0, \infty)$
parameter	$\lambda > 0$
pdf	$\lambda e^{-\lambda x}$
cdf	$1 - e^{-\lambda x}$

- Raised cosine

Notation	$Rcos(x; \mu)$
Suport	$x \in [\mu - 1, \mu + 1]$
parameter	$\mu \in \mathbb{R}$
pdf	$\frac{1}{2}[1 + \cos(\pi(x - \mu))]$

- Poisson

Notation	$Pois(x;\lambda)$
Suport	$x \in \mathbb{N}$
parameter	$\lambda > 0$
pdf	$\frac{\lambda^x}{x!}e^{-\lambda}$

- Binomial

Notation	$Bin(x;n,p)$
Suport	$x \in \{0,1,\ldots,n\}$
parameter	$p \in [0,1], n \in \mathbb{N}$
pdf	$\binom{n}{x}(1-p)^{n-x}p^x$

- Rayleigh

Notation	$Ray(x;\sigma)$
Suport	$x \in [0,\infty)$
parameter	$\sigma \in (0,\infty)$
pdf	$\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$
CDF	$1-e^{-\frac{x^2}{2\sigma^2}}$