Financial Risk: Credit Risk, Lecture 1

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Content of lecture

- Short discussion of the important components of credit risk
- Study different static portfolio credit risk models.
- Discussion of the binomial loss model
- Discussion of the mixed binomial loss model
- Study of a mixed binomial loss model with a beta distribution
- Study of a mixed binomial loss model with a logit-normal distribution
- A short discussion of Value-at-Risk and Expected shortfall

Definition of Credit Risk

Credit risk

— the risk that an obligor does not honor his payments

Example of an obligor:

- A company that have borrowed money from a bank
- A company that has issued bonds.
- A household that have borrowed money from a bank, to buy a house
- A bank that has entered into a bilateral financial contract (e.g an interest rate swap) with another bank.

Example of defaults are

- A company goes bankrupt.
- As company fails to pay a coupon on time, for some of its issued bonds.
- A household fails to pay amortization or interest rate on their loan.

Credit Risk

Credit risk can be decomposed into:

- **arrival risk**, the risk connected to whether or not a default will happen in a given time-period, for a obligor
- **timing risk**, the risk connected to the uncertainness of the exact time-point of the arrival risk (will not be studied in this course)
- **recovery risk**. This is the risk connected to the size of the actual loss if default occurs (will not be studied in this course, we let the recovery be fixed)
- **default dependency risk**, the risk that several obligors jointly defaults during some specific time period. This is one of the most crucial risk factors that has to be considered in a credit portfolio framework.
- The coming two lectures focuses only on **default dependency risk**.

Portfolio Credit Risk is important

- *"Modelling dependence between default events and between credit quality changes is, in practice, one of the biggest challenges of credit risk models".*, David Lando, "Credit Risk Modeling", p. 213.

- *"Default correlation and default dependency modelling is probably the most interesting and also the most demanding open problem in the pricing of credit derivatives. While many single-name credit derivatives are very similar to other non-credit related derivatives in the default-free world (e.g. interest-rate swaps, options), basket and portfolio credit derivative have entirely new risks and features."*,
Philipp Schönbucher, "Credit derivatives pricing models", p. 288.

- *"Empirically reasonable models for correlated defaults are central to the credit risk-management and pricing systems of major financial institutions."*, Darrell Duffie and Kenneth Singleton, "Credit Risk: Pricing, Measurement and Management" , p. 229.

Portfolio Credit Risk is important

- Portfolio credit risk models differ greatly depending on what types of portfolios, and what type of questions that should be considered. For example,
  - models with respect to risk management, such as credit Value-at-Risk (VaR) and expected shortfall (ES)
  - models with respect to valuation of portfolio credit derivatives, such as CDO´s and basket default swaps
- In both cases we need to consider **default dependency risk**, but....
- ...in risk management modelling (e.g. VaR, ES), the timing risk is ignored, and one often talk about **static credit portfolio models**,
- ...while, when pricing credit derivatives, timing risk must be carefully modeled (not treated here)
- The coming two lectures focuses only on **static credit portfolio models**,

Literature

The slides for the coming two lectures are rather self-contained, except for some results taken from Hult & Lindskog.

The content of the lecture today and the next lecture is **partly** based on materials presented in

- Lecture notes by Henrik Hult and Filip Lindskog (Hult & Lindskog) *"Mathematical Modeling and Statistical Methods for Risk Management"*, however, these notes are no longer public available, instead see e.g the book Hult, Lindskog, Hammerlid and Rehn: *"Risk and portfolio analysis - principles and methods"*.

- *"Quantitative Risk Management"* by McNeil A., Frey, R. and Embrechts, P. (Princeton University Press)

- *"Credit Risk Modeling: Theory and Applications"* by Lando, D . (Princeton University Press)

Static Models for homogeneous credit portfolios

- Today we will consider the following static modes for a homogeneous credit portfolio:
  - The binomial model
  - The mixed binomial model
- To understand mixed binomial models, we give a short introduction of conditional expectations
- After this we look at two different mixed binomial models.
- We also shortly discuss Value-at-Risk and Expected shortfall
- Next lecture we consider a mixed binomial model inspired by the Merton framework.

The binomial model for independent defaults

Consider a homogeneous credit portfolio model with m obligors, and where we each obligor can default up to fixed time point, say T. Each obligor have identical credit loss at a default, say ℓ. Here ℓ is a constant.

- Let X_i be a random variable such that

$$X_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults before time } T \\ 0 & \text{otherwise, i.e. if obligor } i \text{ survives up to time } T \end{cases} \quad (1)$$

- We assume that the random variables $X_1, X_2, \ldots X_m$ are **i.i.d**, that is they are all **i**ndependent with **i**dentical **d**istribution.
- Furthermore $\mathbb{P}\left[X_i = 1\right] = p$ so that $\mathbb{P}\left[X_i = 0\right] = 1 - p$.
- The total credit loss in the portfolio at time T, called L_m, is then given by

$$L_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

  thus, N_m is the **number** of defaults in the portfolio up to time T.
- Since ℓ is a constant, we have $\mathbb{P}\left[L_m = k\ell\right] = \mathbb{P}\left[N_m = k\right]$, so it is enough to study the distribution of N_m.

The binomial model for independent defaults, cont.

- Since $X_1, X_2, \dots X_m$ are i.i.d with with $\mathbb{P}[X_i = 1] = p$ we conclude that $N_m = \sum_{i=1}^{m} X_i$ is binomially distributed with parameters m and p, that is $N_m \sim Bin(m, p)$.
- This means that

$$\mathbb{P}[N_m = k] = \binom{m}{k} p^k (1-p)^{m-k}$$

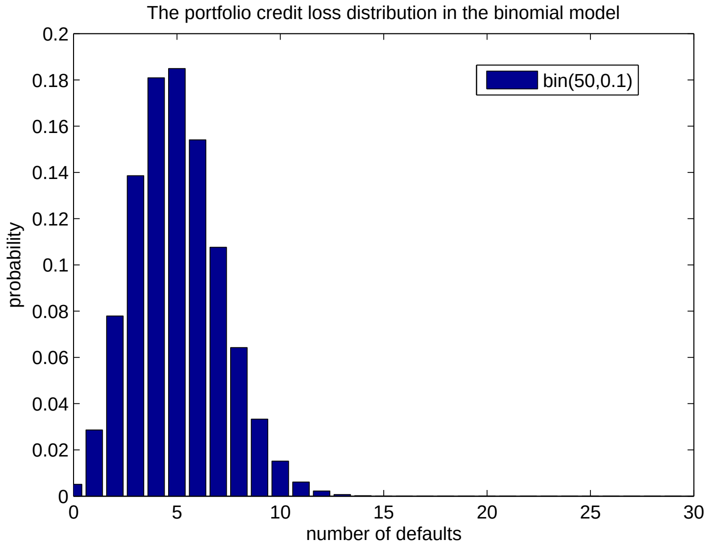
- Recalling the binomial theorem $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$ we see that

$$\sum_{k=0}^{m} \mathbb{P}[N_m = k] = \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} = (p + (1-p))^m = 1$$

  proving that $Bin(m, p)$ is a distribution.
- Furthermore, $\mathbb{E}[N_m] = mp$ since

$$\mathbb{E}[N_m] = \mathbb{E}\left[\sum_{i=1}^{m} X_i\right] = \sum_{i=1}^{m} \mathbb{E}[X_i] = mp.$$

The binomial model for independent defaults, cont.



The binomial model for independent defaults, cont.

- The binomial distribution have very thin "tails", that is, it is extremely unlikely to have many losses (see figure).
- For example, if $p = 5\%$ and $m = 50$ we have that $\mathbb{P}\left[N_m \geq 8\right] = 1.2\%$ and for $p = 10\%$ and $m = 50$ we get $\mathbb{P}\left[N_m \geq 10\right] = 5.5\%$
- The main reason for these small numbers (even for large individual default probabiltes) is due to the independence assumption. To see this, recall that the variance of a random variable $\mathrm{Var}(X)$ measures the degree of the deviation of X around its mean, i.e. $\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right]$.
- Since $X_1, X_2, \ldots X_m$ are independent we have that

$$\mathrm{Var}(N_m) = \mathrm{Var}\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} \mathrm{Var}(X_i) = mp(1-p) \tag{2}$$

where the second equality is due the independence assumption.

The binomial model for independent defaults, cont.

- Furthermore, by Chebyshev's inequality we have that for any random varialbe X, and any $c > 0$ it holds

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq c\right] \leq \frac{\text{Var}(X)}{c^2}$$

- So if $p = 5\%$ and $m = 50$ we have that $\text{Var}(N_m) = 50p(1-p) = 2.375$ and and $\mathbb{E}[N_m] = 50p = 2.5$ implying that having say, 6 more, or less losses than expected, is smaller or equal than 6.6%, since by Chebyshev's inequality

$$\mathbb{P}\left[|N_m - 2.5| \geq 6\right] \leq \frac{2.375}{36} = 6.6\%$$

  Hence, the probability of having a total number of losses outside the interval 2.5 ± 6, i.e. outside the interval $[0, 8.5]$, is smaller than 6.6%.

- In fact, one can show that the deviation of the average number of defaults in the portfolio, $\frac{N_m}{m}$, from the constant p (where $p = \mathbb{E}\left[\frac{N_m}{m}\right]$) goes to zero as $m \to \infty$. Thus, $\frac{N_m}{m}$ converges towards a constant as $m \to \infty$ (the law of large numbers).

Independent defaults and the law of large numbers

- By applying Chebyshev's inequality to the random variable $\frac{N_m}{m}$ together with Equation (2) we get

$$\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq \varepsilon\right] \leq \frac{\mathrm{Var}\left(\frac{N_m}{m}\right)}{\varepsilon^2} = \frac{\frac{1}{m^2}\mathrm{Var}\left(N_m\right)}{\varepsilon^2} = \frac{mp(1-p)}{m^2\varepsilon^2} = \frac{p(1-p)}{m\varepsilon^2}$$

  and we conclude that $\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq \varepsilon\right] \rightarrow 0$ as $m \rightarrow \infty$. Note that this holds for any $\varepsilon > 0$.

- This result is called the weak law of large numbers, and says that the average number of defaults in the portfolio, i.e. $\frac{N_m}{m}$, converges (in probability) to the constant p which is the individual default probability.

- One can also show the so called strong law of large numbers, that is

$$\mathbb{P}\left[\frac{N_m}{m} \rightarrow p \quad \text{when} \quad m \rightarrow \infty\right] = 1$$

  and we say that $\frac{N_m}{m}$ converges almost surely to the constant p. In these lectures we write $\frac{N_m}{m} \rightarrow p$ to indicate almost surely convergence.

Independent defaults lead to unrealistic loss scenarios

- We conclude that the independence assumption, or more generally, the i.i.d assumption for the individual default indicators $X_1, X_2, \dots X_m$ implies that the average number of defaults in the portfolio $\frac{N_m}{m}$ converges to the constant p almost surely.

- Given the recent credit crisis, the assumption of independent defaults is ridiculous. It is an empirical fact, observed many times in the history, that defaults tend to cluster. Hence, the fraction of defaults in the portfolio $\frac{N_m}{m}$ will often have values much bigger than the constant p.

- Consequently, the empirical (i.e. observed) density for $\frac{N_m}{m}$ will have much more "fatter" tails compared with the binomial distribution.

- We will therefore next look at portfolio credit models that can produce more realistic loss scenarios, with densities for $\frac{N_m}{m}$ that have fat tails, and which not implies that the average number of defaults in the portfolio $\frac{N_m}{m}$ converges to a constant with probability 1, when $m \to \infty$.

Conditional expectations

Before we continue this lecture, we need to introduce the concept of conditional expectations

- Let L^2 denote the space of all random variables X such that $\mathbb{E}\left[X^2\right] < \infty$
- Let Z be a random variable and let $L^2(Z) \subseteq L^2$ denote the space of all random variables Y such that $Y = g(Z)$ for some function g and $Y \in L^2$
- Note that $\mathbb{E}\left[X\right]$ is the value μ that minimizes the quantity $\mathbb{E}\left[(X-\mu)^2\right]$. Inspired by this, we define the conditional expectation $\mathbb{E}\left[X \mid Z\right]$ as follows:

Definition of conditional expectations

For a random variable Z, and for $X \in L^2$, the conditional expectation $\mathbb{E}\left[X \mid Z\right]$ is the random variable $Y \in L^2(Z)$ that minimizes $\mathbb{E}\left[(X-Y)^2\right]$.

- Intuitively, we can think of $\mathbb{E}\left[X \mid Z\right]$ as the orthogonal projection of X onto the space $L^2(Z)$, where the scalar product $\langle X, Y\rangle$ is defined as $\langle X, Y\rangle = \mathbb{E}\left[XY\right]$.

Properties of conditional expectations

For a random variable Z it is possible to show the following properties

1. If $X \in L^2$, then $\mathbb{E}[\mathbb{E}[X \mid Z]] = \mathbb{E}[X]$
2. If $Y \in L^2(Z)$, then $\mathbb{E}[YX \mid Z] = Y\mathbb{E}[X \mid Z]$
3. If $X \in L^2$, we define $\text{Var}(X|Z)$ as

$$\text{Var}(X|Z) = \mathbb{E}\left[X^2 \mid Z\right] - \mathbb{E}[X \mid Z]^2$$

   and it holds that $\text{Var}(X) = \mathbb{E}[\text{Var}(X|Z)] + \text{Var}(\mathbb{E}[X \mid Z])$.

Furthermore, for an event A, we can define the conditional probability $\mathbb{P}[A \mid Z]$ as

$$\mathbb{P}[A \mid Z] = \mathbb{E}[1_A \mid Z]$$

where 1_A is the indicator function for the event A (note that 1_A is a random variable). An example: if $X \in \{a, b\}$, let $A = \{X = a\}$, and we get that $\mathbb{P}[X = a \mid Z] = \mathbb{E}\left[1_{\{X=a\}} \mid Z\right]$.

The mixed binomial model

- The binomial model is also the starting point for more sophisticated models. For example, **the mixed binomial model** which randomizes the default probability in the standard binomial model, allowing for stronger dependence.
- The economic intuition behind this randomizing of the default probability $p(Z)$ is that Z should represent some common background variable affecting all obligors in the portfolio.
- **The mixed binomial distribution** works as follows: Let Z be a random variable on $\mathbb{R}$ with density $f_Z(z)$ and let $p(Z) \in [0, 1]$ be a random variable with distribution $F(x)$ and mean $\bar{p}$, that is

$$F(x) = \mathbb{P}\left[p(Z) \leq x\right] \quad \text{and} \quad \mathbb{E}\left[p(Z)\right] = \int_{-\infty}^{\infty} p(z) f_Z(z) dz = \bar{p}. \qquad (3)$$

- Let $X_1, X_2, \ldots X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise. Furthermore, conditional on Z, the random variables $X_1, X_2, \ldots X_m$ are independent and each X_i have default probability $p(Z)$, that is $\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z)$

The mixed binomial model

- Since $\mathbb{P}[X_i = 1 \mid Z] = p(Z)$ we get that $\mathbb{E}[X_i \mid Z] = p(Z)$, because $\mathbb{E}[X_i \mid Z] = 1 \cdot \mathbb{P}[X_i = 1 \mid Z] + 0 \cdot (1 - \mathbb{P}[X_i = 1 \mid Z]) = p(Z)$. Furthermore, note that $\mathbb{E}[X_i] = \bar{p}$ and thus $\bar{p} = \mathbb{E}[p(Z)] = \mathbb{P}[X_i = 1]$ since

$$\mathbb{P}[X_i = 1] = \mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i \mid Z]] = \mathbb{E}[p(Z)] = \int_0^1 p(z) f_Z(z) dz = \bar{p}.$$

  where the last equality is due to (3).

- One can show that

$$\mathrm{Var}(X_i) = \bar{p}(1 - \bar{p}) \quad \text{and} \quad \mathrm{Cov}(X_i, X_j) = \mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 = \mathrm{Var}(p(Z)) \quad (4)$$

- Next, letting all losses be the same and constant given by, say ℓ, then the total credit loss in the portfolio at time T, called $\tilde{L}_m$, is

$$\tilde{L}_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

  thus, N_m is the **number** of defaults in the portfolio up to time T

- Again, since $\mathbb{P}\left[\tilde{L}_m = k\ell\right] = \mathbb{P}[N_m = k]$, it is enough to study N_m.

The mixed binomial model, cont.

- However, since the random variables $X_1, X_2, \ldots X_m$ now only are conditionally independent, given the outcome Z, we have

$$\mathbb{P}\left[N_m = k \,|\, Z\right] = \binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}$$

so since $\mathbb{P}\left[N_m = k\right] = \mathbb{E}\left[\mathbb{P}\left[N_m = k \,|\, Z\right]\right] = \mathbb{E}\left[\binom{m}{k} p(Z)^k (1 - p(Z))^k\right]$ it holds that

$$\mathbb{P}\left[N_m = k\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1 - p(z))^{m-k} f_Z(z) dz. \tag{5}$$

Furthermore, since $X_1, X_2, \ldots X_m$ no longer are independent we have that

$$\mathrm{Var}(N_m) = \mathrm{Var}\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} \mathrm{Var}(X_i) + \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} \mathrm{Cov}(X_i, X_j) \tag{6}$$

and by homogeneity in the model we thus get

$$\mathrm{Var}(N_m) = m\mathrm{Var}(X_i) + m(m-1)\mathrm{Cov}(X_i, X_j). \tag{7}$$

The mixed binomial model, cont.

- So inserting (4) in (7) we get that

$$\mathrm{Var}(N_m) = m\bar{p}(1-\bar{p}) + m(m-1)\left(\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2\right). \tag{8}$$

- Next, it is of interest to study how our portfolio will behave when $m \to \infty$, that is when the number of obligors in the portfolio goes to infinity.
- Recall that $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$ so this and (8) imply that

$$\mathrm{Var}\left(\frac{N_m}{m}\right) = \frac{\mathrm{Var}(N_m)}{m^2} = \frac{\bar{p}(1-\bar{p})}{m} + \frac{(m-1)\left(\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2\right)}{m}.$$

- We therefore conclude that

$$\mathrm{Var}\left(\frac{N_m}{m}\right) \to \mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 \quad \text{as } m \to \infty \tag{9}$$

- Note especially the case when $p(Z)$ is a constant, say p, so that $p = \bar{p}$. Then we are back in the standard binomial loss model and $\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 = p^2 - p^2 = 0$ so $\mathrm{Var}\left(\frac{N_m}{m}\right) \to 0$, i.e. the average number of defaults in the portfolio converge to a constant (which is p) as the portfolio size tend to infinity (this is the law of large numbers.)

The mixed binomial model, cont.

- So in the mixed binomial model, we see from (9) that the law of large numbers do not hold, i.e. $\text{Var}\left(\frac{N_m}{m}\right)$ does not converge to 0.
- Consequently, the average number of defaults in the portfolio, i.e. $\frac{N_m}{m}$, does not converge to a constant as $m \to \infty$.
- This is due to the fact that the random variables $X_1, X_2, \ldots X_m$, are **not** independent. The dependence among the $X_1, X_2, \ldots X_m$, is created by Z.
- However, conditionally on Z, we have that the law of large numbers hold (because if we condition on Z, then $X_1, X_2, \ldots X_m$ are i.i.d with default probability $p(Z)$), that is

$$\text{given a "fixed" outcome of } Z \quad \text{then} \quad \frac{N_m}{m} \to p(Z) \quad \text{as} \quad m \to \infty \tag{10}$$

and since a.s convergence implies convergence in distribution (10) implies that for any $x \in [0, 1]$ we have

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \mathbb{P}\left[p(Z) \leq x\right] \quad \text{when} \quad m \to \infty. \tag{11}$$

The mixed binomial model, cont.

- Note that (11) can also be verified intuitive from (10) by making the following observation. From (10) we have that

$$\mathbb{P}\left[\frac{N_m}{m} \leq \theta \,\middle|\, Z\right] \to \begin{cases} 0 & \text{if } p(Z) > \theta \\ 1 & \text{if } p(Z) \leq \theta \end{cases} \qquad \text{as } m \to \infty$$

that is,

$$\mathbb{P}\left[\frac{N_m}{m} \leq \theta \,\middle|\, Z\right] \to 1_{\{p(Z) \leq \theta\}} \qquad \text{as } \to \infty. \tag{12}$$

- Next, recall that

$$\mathbb{P}\left[\frac{N_m}{m} \leq \theta\right] = \mathbb{E}\left[\mathbb{P}\left[\frac{N_m}{m} \leq \theta \,\middle|\, Z\right]\right] \tag{13}$$

so (12) in (13) renders

$$\mathbb{P}\left[\frac{N_m}{m} \leq \theta\right] \to \mathbb{E}\left[1_{\{p(Z) \leq \theta\}}\right] = \mathbb{P}\left[p(Z) \leq \theta\right] = F(\theta) \qquad \text{as } m \to \infty$$

where $F(x) = \mathbb{P}\left[p(Z) \leq x\right]$, i.e. $F(x)$ is the distribution function of the random variable $p(Z)$.

Large Portfolio Approximation (LPA)

Hence, from the above remarks we conclude the following important result:

Large Portfolio Approximation (LPA) for mixed binomial models

For large portfolios in a mixed binomial model, the distribution of the average number of defaults in the portfolio converges to the distribution of the random variable $p(Z)$ as $m \to \infty$, that is for any $x \in [0,1]$ we have

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \mathbb{P}\left[p(Z) \leq x\right] \quad \text{when} \quad m \to \infty. \tag{14}$$

The distribution $\mathbb{P}\left[p(Z) \leq x\right]$ is called the Large Portfolio Approximation (LPA) to the distribution of $\frac{N_m}{m}$.

The above result implies that if $p(Z)$ has heavy tails, then the random variable $\frac{N_m}{m}$ will also have heavy tails, as $m \to \infty$, which then implies a strong default dependence in the credit portfolio.

The mixed binomial model: the beta function

- One example of a mixing binomial model is to let $p(Z) = Z$ where Z is a beta distribution, $Z \sim \text{Beta}(a, b)$, which can generate heavy tails.
- We say that a random variable Z has beta distribution, $Z \sim \text{Beta}(a, b)$, with parameters a and b, if it's density $f_Z(z)$ is given by

$$f_Z(z) = \frac{1}{\beta(a, b)} z^{a-1}(1-z)^{b-1} \quad a, b > 0, \quad 0 < z < 1 \tag{15}$$

where $\beta(a, b)$ denotes the beta function which satisfies the recursive relation

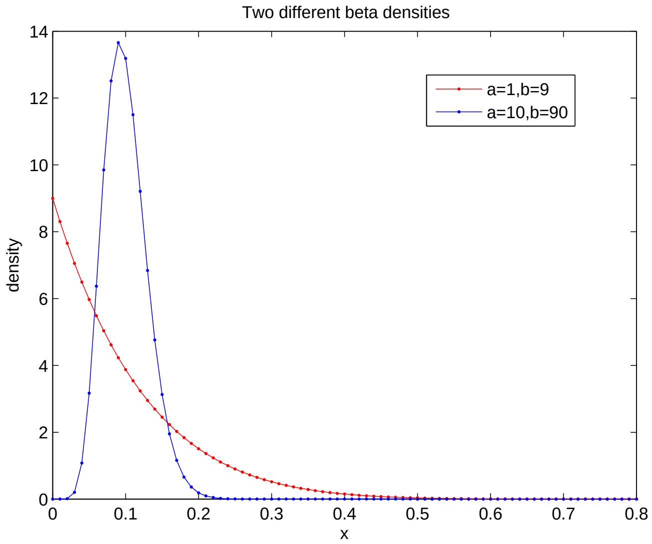
$$\beta(a+1, b) = \frac{a}{a+b}\beta(a, b).$$

Also note that (15) implies that $\mathbb{P}[0 \leq Z \leq 1] = 1$, that is $Z \in [0, 1]$ with probability one.

- Furthermore, since $p(Z) = Z$, the distribution of $\frac{N_m}{m}$ converges to the distribution of the beta distribution, i.e

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \frac{1}{\beta(a, b)} \int_0^x z^{a-1}(1-z)^{b-1} dz \quad \text{as } m \to \infty$$

The mixed binomial model: the beta function, cont.



The mixed binomial model: the beta function, cont.

- If Z has beta distribution with parameters a and b, one can show that

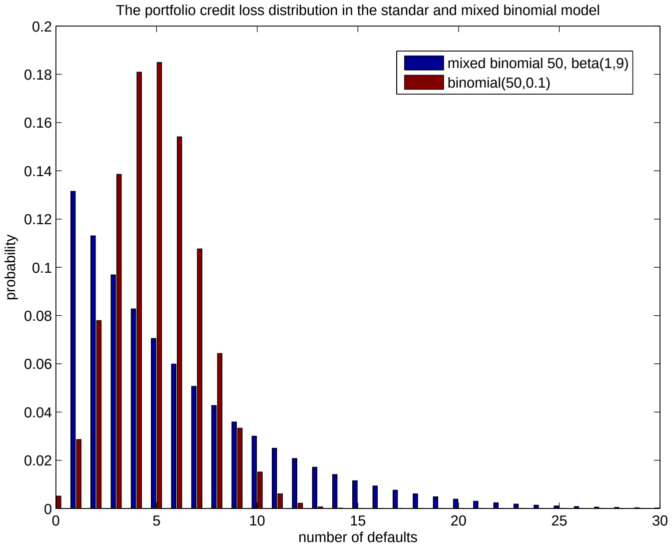
$$\mathbb{E}[Z] = \frac{a}{a+b} \quad \text{and} \quad \mathrm{Var}(Z) = \frac{ab}{(a+b)^2(a+b+1)}.$$

- Consider a mixed binomial model where $p(Z) = Z$ has beta distribution with parameters a and b. Then, by using (5) one can show that

$$\mathbb{P}[N_m = k] = \binom{m}{k} \frac{\beta(a+k, b+m-k)}{\beta(a,b)}. \tag{16}$$

- It is possible to create **heavy tails** in the distribution $\mathbb{P}[N_m = k]$ by choosing the parameters a and b properly in (16). This will then imply more realistic probabilities for extreme loss scenarios, compared with the standard binomial loss distribution (see figure on next page).

The mixed binomial model: the beta function, cont.



Mixed binomial models: logit-normal distribution

- Another possibility for mixing distribution $p(Z)$ is to let $p(Z)$ be a logit-normal distribution. This means that

$$p(Z) = \frac{1}{1 + \exp\left(-(\mu + \sigma Z)\right)}$$

where $\sigma > 0$ and $Z \sim N(0,1)$, that is Z is a standard normal random variable. Note that $p(Z) \in [0,1]$.

- Furthermore, if $0 < x < 1$ then $p^{-1}(x)$ is well defined and given by

$$p^{-1}(x) = \frac{1}{\sigma}\left(\ln\left(\frac{x}{1-x}\right) - \mu\right). \qquad (17)$$

- The mixing distribution $F(x) = \mathbb{P}\left[p(Z) \leq x\right] = \mathbb{P}\left[Z \leq p^{-1}(x)\right]$ for a logit-normal distribution is then given by

$$F(x) = \mathbb{P}\left[Z \leq p^{-1}(x)\right] = N(p^{-1}(x)) \quad \text{for} \quad 0 < x < 1$$

where $p^{-1}(x)$ is given as in Equation (17) and $N(x)$ is the distribution function of a standard normal distribution.

Correlations in mixed binomial models

- Recall the definition of the correlation $\mathrm{Corr}\,(X, Y)$ between two random variables X and Y, given by

$$\mathrm{Corr}\,(X, Y) = \frac{\mathrm{Cov}\,(X, Y)}{\sqrt{\mathrm{Var}\,(X)}\sqrt{\mathrm{Var}\,(Y)}}$$

  where $\mathrm{Cov}(X, Y) = \mathbb{E}\,[XY] - \mathbb{E}\,[X]\,\mathbb{E}\,[Y]$ and $\mathrm{Var}\,(X)) = \mathbb{E}\left[X^2\right] - \mathbb{E}\,[X]^2$.

- Furthermore, also recall that $\mathrm{Corr}\,(X, Y)$ may sometimes be seen as a measure of the "dependence" between the two random variables X and Y.
- Now, let us consider a mixed binomial model as presented previously.
- We are interested in finding $\mathrm{Corr}\,(X_i, X_j)$ for two pairs i, j in the portfolio (by the homogeneous-portfolio assumption this quantity is the same for any pair i, j in the portfolio where $i \neq j$).
- Below, we will therefore for notational convenience simply write ρ_X for the correlation $\mathrm{Corr}\,(X_i, X_j)$.

Correlations in mixed binomial models, cont.

- Recall from previous slides that $\mathbb{P}[X_i = 1 \mid Z] = p(Z)$ where $p(Z)$ is the mixing variable.
- Furthermore, we also now that

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 \quad \text{and} \quad \mathrm{Var}(X_i) = \bar{p}(1-\bar{p}) \tag{18}$$

  where $\bar{p} = \mathbb{E}[p(Z)]$.
- Thus, the correlation ρ_X in a mixed binomial models is then given by

$$\rho_X = \frac{\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2}{\bar{p}(1-\bar{p})} \tag{19}$$

  where $\bar{p} = \mathbb{E}[p(Z)] = \mathbb{P}[X_i = 1]$ is the default probability for each obligor.
- Hence, the correlation ρ_X in a mixed binomial is completely determined by the fist two moments of the mixing variable $p(Z)$, that is $\mathbb{E}[p(Z)]$ and $\mathbb{E}\left[p(Z)^2\right]$.
- Exercise 1: Show that $\mathbb{P}[X_i = 1, X_j = 1] = \mathbb{E}\left[p(Z)^2\right]$ where $i \neq j$.
- Exercise 2: Show that $\mathrm{Var}(X_i) = \mathbb{E}[p(Z)](1 - \mathbb{E}[p(Z)])$.

Value-at-Risk

Recall the definition of Value-at-Risk

Definition of Value-at-Risk

Given a loss L and a confidence level $\alpha \in (0,1)$, then $\mathrm{VaR}_\alpha(L)$ is given by the smallest number y such that the probability that the loss L exceeds y is no larger than $1-\alpha$, that is

$$\begin{aligned}\mathrm{VaR}_\alpha(L) &= \inf\{y \in \mathbb{R} : \mathbb{P}[L > y] \leq 1-\alpha\}\\ &= \inf\{y \in \mathbb{R} : 1 - \mathbb{P}[L \leq y] \leq 1-\alpha\}\\ &= \inf\{y \in \mathbb{R} : F_L(y) \geq \alpha\}\end{aligned}$$

where $F_L(x)$ is the distribution of L.

- Note that Value-at-Risk is defined for a fixed time horizon, so the above definition should also come with a time period, e.g, if the loss L is over one day, then we talk about a one-day $\mathrm{VaR}_\alpha(L)$.
- In credit risk, one typically consider $\mathrm{VaR}_\alpha(L)$ for the loss over one year.
- Note that if $F_L(x)$ is continuous, then $\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha)$

Value-at-Risk for static credit portfolios

- Consider the same type of homogeneous static credit portfolio models as studied previously today, with m obligors and where each obligor can default up to time T. Each obligor have identical credit loss ℓ at a default, where ℓ is a constant.

- The total credit loss in the portfolio at time T is then given by $L_m = \ell N_m$ where N_m is the **number** of defaults in the portfolio up to time T.

- Note that the individual loss ℓ is given by $\tilde{\ell} N$ where N is the notional of the individual loan and $\tilde{\ell}$ is the loss as a fraction of N (i.e $\tilde{\ell} \in [0, 1]$)

- By linearity of VaR (see in lecture notes by H&L) we can without loss of generality assume that $N = 1$, so that $\tilde{\ell} = \ell$, since

$$\mathrm{VaR}_\alpha(cL) = c\mathrm{VaR}_\alpha(L)$$

Value-at-Risk for static credit portfolios, cont.

- If $p(Z)$ is a mixing variable with distribution $F(x)$ we know that

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to F(x) \quad \text{as } m \to \infty$$

which implies that

$$\mathbb{P}\left[L_m \leq x\right] = \mathbb{P}\left[\frac{N_m}{m} \leq \frac{x}{\ell m}\right] \to F\left(\frac{x}{\ell m}\right) \quad \text{as } m \to \infty$$

- Hence, if $F(x)$ is continuous, and if m is "large", we have the following approximation formula for $\mathrm{VaR}_\alpha(L)$

$$\mathrm{VaR}_\alpha(L) \approx \ell \cdot m \cdot F^{-1}(\alpha) \tag{20}$$

where L denotes the loss L_m.

Expected shortfall

The expected shortfall $\mathrm{ES}_\alpha(L)$ is defined as

$$\mathrm{ES}_\alpha(L) = \mathbb{E}\left[\, L \,|\, L \geq \mathrm{VaR}_\alpha(L)\right]$$

and one can show that

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)du.$$

Hence, for the same static credit portfolio as on the two previous slides, we have the following approximation formula for $\mathrm{ES}_\alpha(L)$ (when m is large)

$$\mathrm{ES}_\alpha(L) \approx \frac{\ell \cdot m}{1-\alpha}\int_\alpha^1 F^{-1}(u)du$$

where L denotes the loss L_m and where we used (20). Here, $F(x)$ is the continuous distribution of the mixing variable $p(Z)$.

Thank you for your attention!