Financial Risk
2-rd quarter 2012/13
Tuesdays 10.15 – 12.00 and Thursday 13.15 – 15.00 in MVF31 and Pascal, resp.

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

This course:

- Learn some fun things about financial risks
- Learn some basic risk management tools from Extreme Value Statistics
- See some basic quantitative Credit Risk models

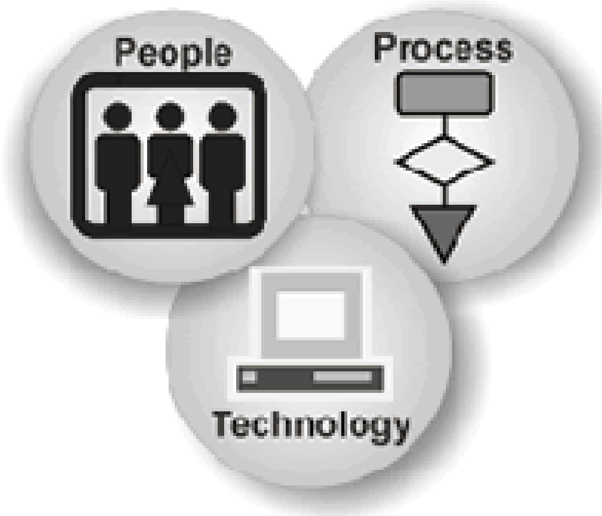
But not

- A complete systematic account of financial risk management
- Financial time series modeling
- Black-Scholes option pricing methods
- Macroeconomics

and

There are risks which cannot be handled by mathematical models!!

- Credit risk
- Market risk
- Operational risk
- Insurance risk
- Liquidity risk
- Reputational risk
- Legal risk
- and so on …



Operational risk

Risk factors

L= "-P/L" = Loss – Profit = $f(X_1, \ldots, X_d)$, $X_1, \ldots, X_d$ risk factors, *e.g.* exchange rates, interest rates, index movements, stock prizes, ….

Example: Linear portfolio, α_i shares of stock i , stock prize $S_{t,i}$

$$L = -\sum_{i=1}^d \alpha_i S_{t+1,i} + \sum_{i=1}^d \alpha_i S_{t,i} = -\sum_{i=1}^d \alpha_i S_{t,i}\left(\frac{S_{t+1,i} - S_{t,i}}{S_{t,i}}\right)$$

How big is the risk?

Quantitative risk management methods:

- Historical data or historical simulation
- Stress testing ("scenarios")
- Sensitivity measures ("the greeks")

- Full statistical modeling (often multivariate normal + linear portfolio)
- Semiparametric modeling of the "tails" of the *loss-profit* distribution (univariate Extreme Value Statistics) ***(this course)***
- Semiparametric modeling of the tails of the multivariate distribution of the risk factors (multivariate Extreme Value Statistics, "Copulas") + computation of the *loss-profit* distribution analytically or via stochastic simulation

How **big** is the risk??

Don’t look at the stars with a microscope --- and don’t use statistical methods tailored to means and typical behavior to study extreme occurrences:

Use Extreme Value Statistics! (if not -- you will not see the important things)

Normal statistics: based on the normal distribution, concerned with averages and typical behavior

Extreme Value Statistics: based on Extreme Value and Generalized Pareto Distributions, concerned with rare events

Model checking and adjustment

- **Stationarity**
- Independence
- Distributional fit
- Period and threshold choice

Statistical model ⇄ Data analysis

- Choice of distributions
- Estimation (often of quantiles)
- Confidence intervals
- “Prediction intervals”
- Hypothesis testing
- Regression and dependence modeling ***(not this course)***
- Prediction ***(not this course)***
- ***Understanding!!***

refresh your basic statistics knowledge!

Basic EVS:
--- Block Maxima: EV (= GEV) distribution for maxima
--- Peaks over Thresholds: GP distribution for tails

Why?

- **stability:** maxima of variables which are EV distributed are also EV; going to higher levels preserves the GP distribution of exceedances (cf. "standard statistics: sums of normally distributed variables have a normal distribution)

- **asymptotics:** maxima of many independent variables are often (approximately) EV distributed; asymptotically tails are GP when maxima are EV (cf. "standard statistics: sums of many small "errors" are often (approximately) normally distributed the "central limit theorem")

- ”transition”: easy to go back and forth between GP and EV

but don't believe in models blindly

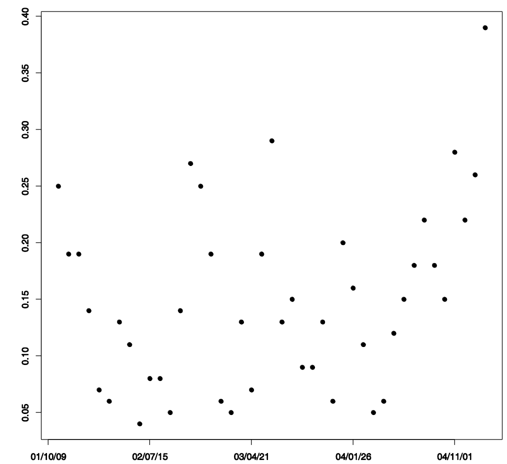
The Block Maxima method *(Coles p. 45-53)*

the Extreme Value (EV) distribution:

$$G(x) = \exp\{-(1+\gamma\tfrac{x-\mu}{\sigma})_+^{-1/\gamma}\}$$

the special case $\gamma = 0$ is the Gumbel distribution

$$G_0(x) = \exp\{-\exp\{-\tfrac{x-\mu}{\sigma}\}\}$$



Maximum long term US interest rate, constant maturity, nominal 1 month, percentage points

Fit to the measured block (=weekly, or monthly, or yearly, or …) maxima, assuming independence, using maximum likelihood. Find *p*-th quantile from top (= value such that the risk that it is exceeded is 100*p* %) by solving

$$\hat{\bar{G}}(x_p) := 1 - \hat{G}(x_p) = p,$$

where ^ means that parameters are replaced by their estimated values.

confidence intervals via profile likelihood or delta method *(later lectures)*

Some mathematics behind the Block Maxima Method:

$X_1, X_2, \ldots$ independent identically distributed random (i.i.d.) variables with distribution function (d.f.) F (*e.g.* daily interest rates, c.f. previous slide)

$M_n = \max\{X_1, \ldots X_n\}$ = the maximum of the first n variables (*e.g. n* = 30 and X daily interest rates gives plot on previous slide)

$$\begin{aligned} P\big(M_n \leq x\big) &= P\big(X_1 \leq x, \ldots X_n \leq x\big) \\ &= P\big(X_1 \leq x\big) \times \ldots \times P\big(X_n \leq x\big) = F(x)^n \end{aligned}$$

Exercise: Show that the EV distributions are *max-stable*, *i.e.* that the maximum of n i.i.d. EV-distributed variables also have an EV distribution, *i.e.* that if $G(x) = \exp\{-(1+\gamma\frac{x-\mu}{\sigma})_+^{-1/\gamma}\}$, then

$$G(x)^n = \exp\{-(1+\gamma\frac{(x-\mu_n)/\sigma_n-\mu}{\sigma})^{-1/\gamma}\} = G(\frac{x-\mu_n}{\sigma_n})$$

and find $\mu_n, \ \sigma_n.$

Theorem: *The distribution function G is max-stable if and only if it is an EV distribution*

In the previous exercise it was shown that the EV distributions are max-stable, which proves half of this theorem. The other half consists of solving the functional equations

$$G(x)^n = G(\tfrac{x-\mu_n}{\sigma_n}), \text{ for } n = 1, 2, \ldots$$

to find that the EV distributions are the only solutions.

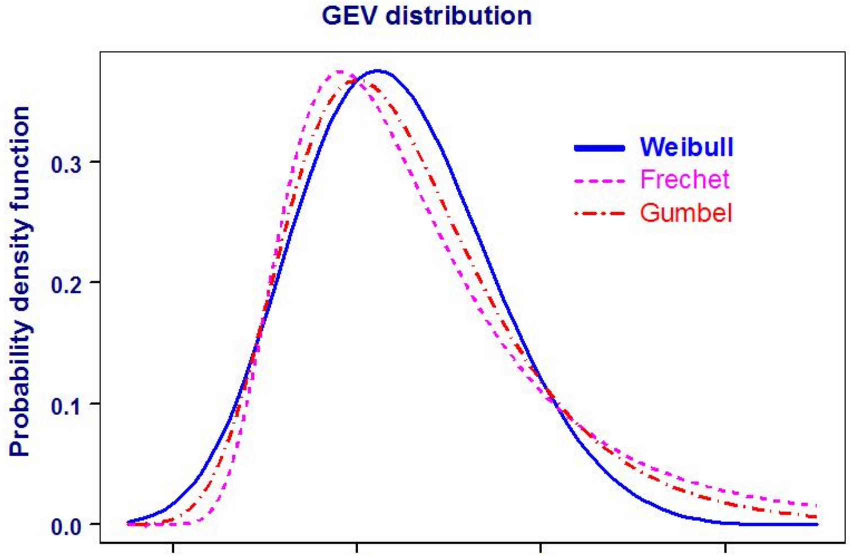
Theorem: *If there are constants $b_n > 0, a_n$ such that*

$$P\Big(\tfrac{M_n - a_n}{b_n} \leq x\Big) \to G(x), \text{ as } n \to \infty \text{ for all } x$$

then G(x) is an EV distribution. ***(cf. the central limit theorem)***

This is proved by showing that it follows that *G(x)* must be max-stable

Challenge: Prove this!



$$g(x) = \frac{d}{dx}G(x) = \frac{1}{\sigma}(1+\gamma\frac{x-\mu}{\sigma})_+^{-1/\gamma-1}\exp\{-(1+\gamma\frac{x-\mu}{\sigma})_+^{-1/\gamma}\}$$

Densities of the generalized Extreme Value distribution

$\gamma > 0$ Frechet distribution, finite left endpoint of distribution: $x > \mu + \frac{\sigma}{\gamma}$

$\gamma = 0$ Gumbel distribution, unbounded support

$\gamma < 0$ Weibull distribution, finite right endpoint of distribution: $x < \mu + \frac{\sigma}{|\gamma|}$

the distribution is "heavytailed" for $\gamma > 0$: then moments of order greater than $1/\gamma$ are infinite/don't exist; thus if $\gamma > 1/2$ then the variance doesn't exist, if $\gamma > 1$ then the mean doesn't exist either

Exercise: For each of the following three distributions, show that, with the given norming constants $a_n,\ b_n$, there is a d.f. *G(x)* such that

$$P(\frac{M_n - a_n}{b_n} \le x) \to G(x), \quad \text{as} \;\; n \to \infty \;\; \text{for all} \;\; x,$$

and find this *G(x)*.

1.) The exponential distribution $F(x) = 1 - e^{-\lambda x}, \quad \lambda > 0, \quad x \ge 0$
(use $a_n = \frac{log n}{\lambda}, \quad b_n = 1/\lambda$)

2.) The Pareto distribution $F(x) = 1 - (\frac{\kappa}{\kappa + x})^{\alpha}, \quad \kappa, \alpha > 0, \quad x \ge 0$
(use $a_n = \kappa n^{1/\alpha} - \kappa, \quad b_n = \kappa n^{1/\alpha}/\alpha$)

3.) The uniform distribution $F(x) = x, \quad 0 \le x \le 1$
(use $a_n = 1 - 1/n, \quad b_n = 1/n$)

(A perhaps unnecessary explanation) **What does**

$$P\left(\frac{M_n-a_n}{b_n} \le x\right) \to G(x), \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x,$$

mean in practice? That $P\left(\frac{M_n-a_n}{b_n} \le x\right) \approx G(x)$, for large n, or,

with $y = b_n x + a_n$ and $G(x) = \exp\{-(1+\gamma\frac{x-\mu'}{\sigma'}))^{-1/\gamma}\}$, that

$$\begin{aligned} P(M_n \le y) &\approx G\left(\frac{y-a_n}{b_n}\right) = \exp\{-(1+\gamma\frac{y-(a_n+b_n\mu')}{b_n\sigma'})^{-1/\gamma}\} \\ &= \exp\{-(1+\gamma\frac{y-\mu}{\sigma})^{-1/\gamma}\}, \text{ for } \mu = a_n + b_n\mu',\ \sigma = b_n\sigma. \end{aligned}$$

Since all the parameters are unknown anyway, we are left with the problem of estimating μ, σ from data, *i.e.* with the Block Maxima method.