Financial Risk: Credit Risk, Lecture 2

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November 15, 2012

Content of Lecture

- Discussion of a mixed binomial model inspired by the Merton model
- Derive the large-portfolio approximation formula in this framework
- Discussion of a mixed binomial model where the factor has discrete distribution.

The mixed binomial model inspired by the Merton Model

- Consider a credit portfolio model, not necessary homogeneous, with m obligors, and where each obligor can default up to fixed time point, say T.
- Assume that each obligor i (think of a firm named i) follows the Merton model, in the sense that obligor i-s assets $V_{t,i}$ follows the dynamics

$$dV_{t,i} = rV_{t,i}dt + \sigma_i V_{t,i} dB_{t,i} \tag{1}$$

where $B_{t,i}$ is a stochastic process defined as

$$B_{t,i} = \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i}. \tag{2}$$

Here $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ are independent standard Brownian motions

- It is then possible to show that $B_{t,i}$ is also a standard Brownian motion. Hence, due to (1) we then know that $V_{t,i}$ is a GBM so by using Ito´s lemma, we get

$$V_{t,i} = V_{0,i} e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma_i B_{t,i}}$$

The mixed binomial model inspired by the Merton Model

- The intuition behind (1) and (2) is that the asset for each obligor i is driven by a common process $W_{t,0}$ representing the economic environment, and an individual process $W_{t,i}$ unique for obligor i, where $i = 1, 2, \ldots, m$.
- This means that the asset for each obligor i, depend both on a macroeconomic random process (common for all obligors) and an idiosyncratic random process (i.e. unique for each obligor). This will create a **dependence** among these obligors. To see this, recall that $\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\,\mathbb{E}[X_j]$ so due to (2)

$$
\begin{aligned}
\mathrm{Cov}\left(B_{t,i}, B_{t,j}\right) &= \mathbb{E}\left[B_{t,i}B_{t,j}\right] - \mathbb{E}\left[B_{t,i}\right]\mathbb{E}\left[B_{t,j}\right] \\
&= \mathbb{E}\left[\left(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,i}\right)\left(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,j}\right)\right] \\
&= \mathbb{E}\left[\rho W_{t,0}^2\right] + \sqrt{\rho}\sqrt{1-\rho}\left(\mathbb{E}\left[W_{t,0}W_{t,i}\right] + \mathbb{E}\left[W_{t,0}W_{t,j}\right]\right) \\
&\qquad + (1-\rho)\,\mathbb{E}\left[W_{t,j}W_{t,i}\right] \\
&= \rho\mathbb{E}\left[W_{t,0}^2\right] = \rho t
\end{aligned}
$$

where the third equality is due to $\mathbb{E}\left[W_{t,j}W_{t,i}\right] = 0$ when $i \neq j$.

The mixed binomial model inspired by the Merton Model

- Hence, $\mathrm{Cov}\left(B_{t,i}, B_{t,j}\right) = \rho t$ which implies that there is a dependence of the processes that drives the asset values $V_{t,i}$. To be more specific,

$$\mathrm{Corr}\left(B_{t,i}, B_{t,j}\right) = \frac{\mathrm{Cov}\left(B_{t,i}, B_{t,j}\right)}{\sqrt{\mathrm{Var}\left(B_{t,i}\right)}\sqrt{\mathrm{Var}\left(B_{t,i}\right)}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho \tag{3}$$

  so $\mathrm{Corr}\left(B_{t,i}, B_{t,j}\right) = \rho$ which is the mutual dependence among the obligors created by the macroeconomic latent variable $W_{t,0}$

- Note that if $\rho = 0$, we have $\mathrm{Corr}\left(B_{t,i}, B_{t,j}\right) = 0$ which makes the asset values $V_{t,1}, V_{t,2}, \ldots, V_{t,m}$ independent (so the obligors are independent).

- Next, let D_i be the debt level for each obligor i and recall from the Merton model that obligor i defaults if $V_{T,i} \leq D_i$, that is if

$$V_{0,i} e^{\left(r - \frac{1}{2}\sigma_i^2\right)T + \sigma_i B_{T,i}} < D_i \tag{4}$$

  which, by using the definition of $B_{t,i}$ is equivalent with the event

$$\ln V_{0,i} - \ln D_i + \left(r - \frac{1}{2}\sigma_i^2\right)T + \sigma_i \left(\sqrt{\rho} W_{T,0} + \sqrt{1-\rho} W_{T,i}\right) < 0 \tag{5}$$

The mixed binomial model inspired by the Merton Model

- Next, recall that for each i, $W_{i,T} \sim N(0, T)$, i.e $W_{i,T}$ is normally distributed with zero mean and variance T. Hence, if $Y_i \sim N(0,1)$, $W_{i,T}$ has the same distribution as $\sqrt{T} Y_i$ for $i = 0, 1, \ldots, m$ where $Y_0, Y_1, \ldots, Y_M$ also are independent. Furthermore, define Z as Y_0, i.e $Z = Y_0$. This in (5) yields

$$\ln V_{0,i} - \ln D_i + (r - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}\sqrt{T} Z + \sqrt{1-\rho}\sqrt{T} Y_i \right) < 0 \quad (6)$$

and dividing with $\sigma_i \sqrt{T}$ renders

$$\frac{\ln V_{0,i} - \ln D_i + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}} + \sqrt{\rho} Z + \sqrt{1-\rho} Y_i < 0. \quad (7)$$

We can rewrite the inequality (7) as

$$Y_i < \frac{-\left(C_i + \sqrt{\rho} Z\right)}{\sqrt{1-\rho}} \quad (8)$$

where C_i is a constant given by

$$C_i = \frac{\ln\left(V_{0,i}/D_i\right) + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}} \quad (9)$$

The mixed binomial model inspired by the Merton Model

- Hence, from the previous slides we conclude that

$$V_{T,i} < D_i \quad \text{is equivalent with} \quad Y_i < \frac{-\left(C_i + \sqrt{\rho} Z\right)}{\sqrt{1-\rho}} \tag{10}$$

  where C_i is a constant given by (9).

- Next define X_i as

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D_i \\ 0 & \text{if } V_{T,i} > D_i \end{cases} \tag{11}$$

- Then (10) implies that

$$\begin{aligned} \mathbb{P}\left[X_i = 1 \,|\, Z\right] = \mathbb{P}\left[V_{T,i} < D_i \,|\, Z\right] &= \mathbb{P}\left[Y_i < \frac{-\left(C_i + \sqrt{\rho} Z\right)}{\sqrt{1-\rho}} \,\middle|\, Z \right] \\ &= N\left(\frac{-\left(C_i + \sqrt{\rho} Z\right)}{\sqrt{1-\rho}} \right) \end{aligned} \tag{12}$$

  where $N(x)$ is the distribution function of a standard normal distribution.

- The last equality in (12) follows from the fact that $Y_i \sim N(0,1)$ and that Y_i is independent of Z in (10).

The mixed binomial model inspired by the Merton Model

- Next, assume that all obligors in the model are identical, so that $V_{0,i} = V_0$, $D_i = D$ and thus $C_i = C$ for $i = 1, 2, \ldots, m$.
- Then we have a homogeneous static credit portfolio, where we consider the time period up to T.
- Furthermore, Equation (12) implies that

$$\mathbb{P}\left[X_i = 1 \,|\, Z\right] = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{13}$$

  where C is a constant given by (9) with $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$ and thus $C_i = C$ for all obligors i.
- Let Z be the "economic background variable" in our homogeneous portfolio and define $p(Z)$ as

$$p(Z) = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{14}$$

  where $N(x)$ is the distribution function of a standard normal distribution.

The mixed binomial model inspired by the Merton Model

- Since, $p(Z) \in [0,1]$, we would like to use $p(Z)$ in a mixed binomial model.

- To be more specific, let $X_1, X_2, \ldots X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise.

- Furthermore, conditional on Z, the random variables $X_1, X_2, \ldots X_m$ are independent and each X_i have default probability $p(Z)$, that is

$$\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z) = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right). \tag{15}$$

- We call this the mixed binomial model inspired by the Merton model or sometimes simply a mixed binomial Merton model.

The mixed binomial Merton model

- Let $\tilde{L}_m = \sum_{i=1}^{m} \ell X_i$ denote the total credit loss in our portfolio at time T. We now want to study $\mathbb{P}\left[\tilde{L}_m \leq x\right]$ in our portfolio where X_i, conditional on Z, have default probabilities $p(Z)$ given by (15).
- Since the portfolio is homogeneous, all losses are the same and constant given by, say ℓ, so

$$\tilde{L}_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

  thus, N_m is the **number** of defaults in the portfolio up to time T. Hence, since $\mathbb{P}\left[\tilde{L}_m = k\ell\right] = \mathbb{P}\left[N_m = k\right]$, it is enough to study $\mathbb{P}\left[N_m \leq n\right]$ where $n = 0, 1, 2 \ldots, m$ instead of $\mathbb{P}\left[\tilde{L}_m \leq x\right]$.
- Next, note that $\mathbb{P}\left[N_m \leq n\right] = \sum_{k=0}^{n} \mathbb{P}\left[N_m = k\right]$ and

$$\mathbb{P}\left[N_m = k\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1 - p(z))^{m-k} f_Z(z) dz \qquad (16)$$

  where $f_Z(z)$ is the density of Z.

The mixed binomial Merton model, cont.

- In our case Z is a standard normal random variable so

$$\mathbb{P}\left[N_m = k\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(u)^k (1 - p(u))^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \qquad (17)$$

Furthermore, $p(u)$ is given by $p(u) = N\left(\frac{-(C+\sqrt{\rho}u)}{\sqrt{1-\rho}}\right)$ where $N(x)$ is the distribution function of a standard normal distribution.

- Hence, $\mathbb{P}\left[N_m \leq n\right]$ is given by

$$\mathbb{P}\left[N_m \leq n\right] = \sum_{k=0}^{n} \binom{m}{k} \int_{-\infty}^{\infty} N\left(\frac{-\left(C+\sqrt{\rho}u\right)}{\sqrt{1-\rho}}\right)^k \cdot \left(1 - N\left(\frac{-\left(C+\sqrt{\rho}u\right)}{\sqrt{1-\rho}}\right)\right)^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \qquad (18)$$

Mixed binomial Merton: Large Portfolio Approx. (LPA)

- So if we know C (later we show how to find C) we can therefore find $\mathbb{P}[N_m \leq n]$ by numerically evaluate the expression in the RHS in (18).
- However, there is another way to find a very convenient approximation of $\mathbb{P}[N_m \leq n]$.
- To see this, recall from the last lecture that in any mixed binomial distribution we have that

$$\mathbb{P}\left[\frac{N_m}{m} \leq \theta\right] \to F(\theta) \quad \text{as } m \to \infty \tag{19}$$

where $F(x)$ is the distribution function of $p(Z)$, i.e. $F(x) = \mathbb{P}[p(Z) \leq x]$

- But for any x we then have

$$\mathbb{P}[N_m \leq x] = \mathbb{P}\left[\frac{N_m}{m} \leq \frac{x}{m}\right] \approx F\left(\frac{x}{m}\right) \quad \text{if } m \text{ is "large"}.$$

- Hence, we can approximate $\mathbb{P}[N_m \leq n]$ with $F\left(\frac{n}{m}\right)$ instead of numerically compute the quite involved expression in the RHS in (18).

The mixed binomial Merton model and LPA, cont.

- We therefore next want to find an explicit expression of $F(\theta)$ where $F(\theta) = \mathbb{P}\left[p(Z) \leq \theta\right]$. From (15) we know that $p(Z) = N\left(\frac{-(C+\sqrt{\rho}Z)}{\sqrt{1-\rho}}\right)$ where Z is a standard normal random variable, i.e. $Z \sim N(0,1)$.
- Hence, $F(\theta) = \mathbb{P}\left[p(Z) \leq \theta\right] = \mathbb{P}\left[N\left(\frac{-(C+\sqrt{\rho}Z)}{\sqrt{1-\rho}}\right) \leq \theta\right]$ so

$$
\begin{aligned}
\mathbb{P}\left[N\left(\frac{-\left(C+\sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq \theta\right] &= \mathbb{P}\left[\frac{-\left(C+\sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \leq N^{-1}(\theta)\right] \\
&= \mathbb{P}\left[-Z \leq \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta)+C\right)\right] \\
&= N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta)+C\right)\right)
\end{aligned}
$$

where the last equality is due to $\mathbb{P}\left[-Z \leq x\right] = \mathbb{P}\left[Z \geq -x\right] = 1 - \mathbb{P}\left[Z \leq -x\right]$ and $1 - N(-x) = N(x)$ for any x, due to the symmetry of a standard normal random variable.

The mixed binomial Merton model and LPA, cont.

- Hence, $F(\theta) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta) + C\right)\right)$ so what is left is to find C.
- Since our model is inspired by the Merton model, we have that

$$X_i = \begin{cases} 1 & \text{if } V_T < D \\ 0 & \text{if } V_T > D \end{cases} \tag{20}$$

so $\mathbb{P}[X_i = 1] = \mathbb{P}[V_T < D]$. However, from (7) and (10) we conclude that

$$V_T < D \quad \Leftrightarrow \quad \sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C \tag{21}$$

where C is given by Equation (9) in the homogeneous case where $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$ and consequently $C_i = C$ for $i = 1, 2, \ldots, m$.

Furthermore, since Z and Y_i are standard normals then $\sqrt{\rho}Z + \sqrt{1-\rho}Y_i$ will also be standard normal. Hence, $\mathbb{P}\left[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C\right] = N(-C)$ and this observation together with (21) implies that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[V_T < D] = N(-C). \tag{22}$$

- Recall that $\bar{p} = \mathbb{E}\left[p(Z)\right] = \int_0^1 p(z) f_Z(z) dz$ so $\bar{p} = \mathbb{P}\left[X_i = 1\right]$ since $\mathbb{P}\left[X_i = 1 \mid Z\right] = p(Z)$ and thus

$$\mathbb{P}\left[X_i = 1\right] = \mathbb{E}\left[\mathbb{P}\left[X_i = 1 \mid Z\right]\right] = \mathbb{E}\left[p(Z)\right] = \bar{p}$$

- Hence, from (22) we have $\bar{p} = N\left(-C\right)$ so

$$C = -N^{-1}\left(\bar{p}\right) \tag{23}$$

which means that we can ignore C (and thus also ignore V_0, D, σ and r, see (9)) and instead directly work with the default probability $\bar{p} = \mathbb{P}\left[X_i = 1\right]$. Hence, we estimate $\bar{p}$ to 5%, say, which then implicitly defines the quantizes V_0, D, σ and r via (9) and (23).

- Finally, going back to $F(\theta) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta) + C\right)\right)$ and using (23) we conclude that

$$F(\theta) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta) - N^{-1}\left(\bar{p}\right)\right)\right) \tag{24}$$

where $F(\theta) = \mathbb{P}\left[p(Z) \leq \theta\right]$.

The mixed binomial Merton model and LPA, cont.

- Hence, if m is large enough, we can in a mixed binomial model inspired by the Merton model, do the following approximation of the portfolio loss probability $\mathbb{P}\left[N_m \leq n\right] = \mathbb{P}\left[\frac{N_m}{m} \leq \frac{n}{m}\right] \approx F\left(\frac{n}{m}\right)$, that is

$$\mathbb{P}\left[N_m \leq n\right] \approx N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}\left(\frac{n}{m}\right) - N^{-1}\left(\bar{p}\right)\right)\right). \qquad (25)$$

  where $\bar{p} = \mathbb{P}\left[X_i = 1\right]$ is the individual default probability for each obligor.

- The approximation (24) or equivalently (25), is sometimes denoted the **LPA** in a static Merton framework, and was first introduced by Vasicek 1991, at KMV, in the paper *"Limiting loan loss probability distribution"*.

- The **LPA** in a Merton framework and its offsprings (i.e. variants) is today **widely** used in the industry (Moody's-KMV, CreditMetrics etc. etc.) for risk management of large credit/loan portfolios, especially for computing regulatory capital in **Basel II** and **Basel III** (Basel III is to be implemented before end of 2013).

The mixed binomial Merton model: The role of ρ

- Recall from (3), that ρ was the correlation parameter describing the dependence between the Brownian motions $B_{t,i}$ that drives each obligor i's asset price, i.e. $\mathrm{Cov}(B_{t,i}, B_{t,j}) = \rho t$ so that $\mathrm{Corr}(B_{t,i}, B_{t,j}) = \rho$.
- Since $X_i = 1_{\{V_{T,i} \leq D\}}$ we know that X_i and X_j are **dependent** because $\mathrm{Cov}(B_{t,i}, B_{t,j}) = \rho t$ where $\rho \neq 0$. Furthermore, if $\rho \neq 0$ it generally holds that $\mathrm{Cov}(X_i, X_j) \neq 0$ since

$$
\begin{aligned}
\mathrm{Cov}(X_i, X_j) &= \mathbb{E}\left[1_{\{V_{T,i} \leq D\}} 1_{\{V_{T,j} \leq D\}}\right] - \mathbb{E}\left[1_{\{V_{T,i} \leq D\}}\right] \mathbb{E}\left[1_{\{V_{T,j} \leq D\}}\right] \\
&= \mathbb{P}\left[V_{T,i} \leq D, V_{T,j} \leq D\right] - \mathbb{P}\left[V_{T,i} \leq D\right] \mathbb{P}\left[V_{T,j} \leq D\right] \\
&= \mathbb{P}\left[V_{T,i} \leq D, V_{T,j} \leq D\right] - \bar{p}^2
\end{aligned}
\tag{26}
$$

and $\mathbb{P}\left[V_{T,i} \leq D, V_{T,j} \leq D\right] \neq \bar{p}^2$ since $\mathrm{Cov}(B_{t,i}, B_{t,j}) = \rho t$ with $\rho \neq 0$ implies (see also Equation (21) and (22))

$$
\begin{aligned}
\mathbb{P}\left[V_{T,i} \leq D, V_{T,j} \leq D\right] &= \mathbb{P}\left[B_{T,i} < -\sqrt{T}C, B_{T,j} < -\sqrt{T}C\right] \\
&\neq \mathbb{P}\left[B_{T,i} < -\sqrt{T}C\right] \mathbb{P}\left[B_{T,j} < -\sqrt{T}C\right] = \bar{p}^2.
\end{aligned}
$$

Hence, $\mathrm{Cov}(X_i, X_j) \neq 0$ when $\rho \neq 0$.

The mixed binomial Merton model: The role of ρ, cont.

- Next, assume that $\rho = 0$ so that $\mathrm{Cov}(B_{t,i}, B_{t,j}) = 0$. Furthermore, by (2) we have that $B_{t,i} = W_{t,i}$ when $\rho = 0$ since

$$B_{t,i} = \sqrt{0} W_{t,0} + \sqrt{1-0} W_{t,i} = W_{t,i} \tag{27}$$

where $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ are **independent** standard Brownian motions.

- Equation (27) and the **independence** among $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ then imply

$$\begin{aligned}
\mathbb{P}\left[V_{T,i} \leq D, V_{T,j} \leq D\right] &= \mathbb{P}\left[B_{T,i} < -\sqrt{T}C, B_{T,j} < -\sqrt{T}C\right] \\
&= \mathbb{P}\left[W_{T,i} < -\sqrt{T}C, W_{T,j} < -\sqrt{T}C\right] \\
&= \mathbb{P}\left[W_{T,i} < -\sqrt{T}C\right]\mathbb{P}\left[W_{T,j} < -\sqrt{T}C\right] \\
&= \mathbb{P}\left[V_{T,i} \leq D\right]\mathbb{P}\left[V_{T,j} \leq D\right] = \bar{p}^2
\end{aligned}$$

and plugging this into (26) yields that $\mathrm{Cov}(X_i, X_j) = 0$.

The mixed binomial Merton model: The role of ρ, cont.

- From the above studies we conclude that

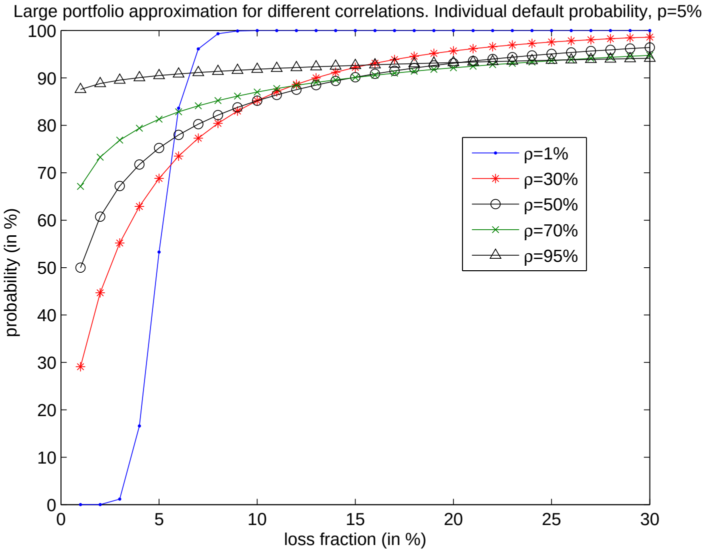
$$\text{Cov}(X_i, X_j) = 0 \quad \text{if } \rho = 0 \tag{28}$$

and

$$\text{Cov}(X_i, X_j) \neq 0 \quad \text{if } \rho \neq 0. \tag{29}$$

- We therefore conclude that ρ is a measure of **default dependence** among the zero-one variables $X_1, X_2, \ldots, X_m$ in the mixed binomial Merton model.

The mixed Merton binomial model and LPA



The mixed Merton binomial model and LPA, cont.

- Given the limiting distribution $F(\theta)$

$$F(\theta) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(\theta) - N^{-1}(\bar{p})\right)\right) \tag{30}$$

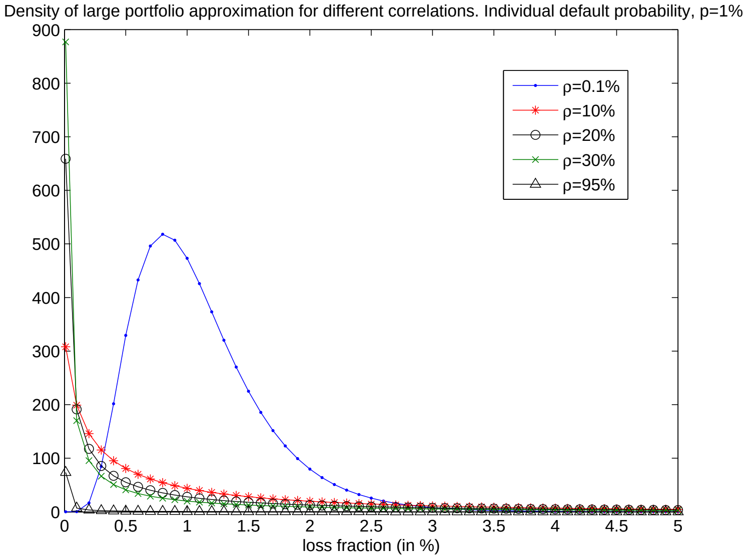
we can also find the density $f_{\text{LPA}}(\theta)$ of $F(\theta)$, that is $f_{\text{LPA}}(\theta) = \frac{dF(\theta)}{d\theta}$.

- It is possible to show that

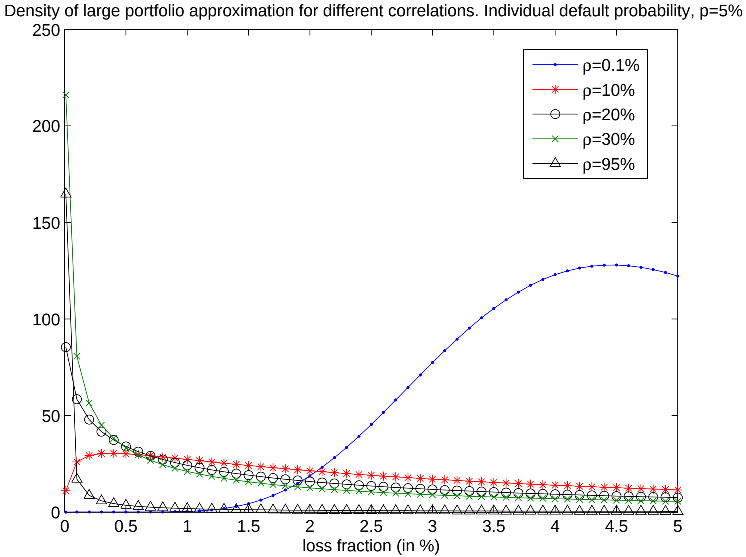
$$f_{\text{LPA}}(\theta) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(\frac{1}{2}(N^{-1}(\theta))^2 - \frac{1}{2\rho}\left(N^{-1}(\bar{p}) - \sqrt{1-\rho}N^{-1}(\theta)\right)^2\right) \tag{31}$$

- This density is just an approximation, and fails for small number of the loss fraction.

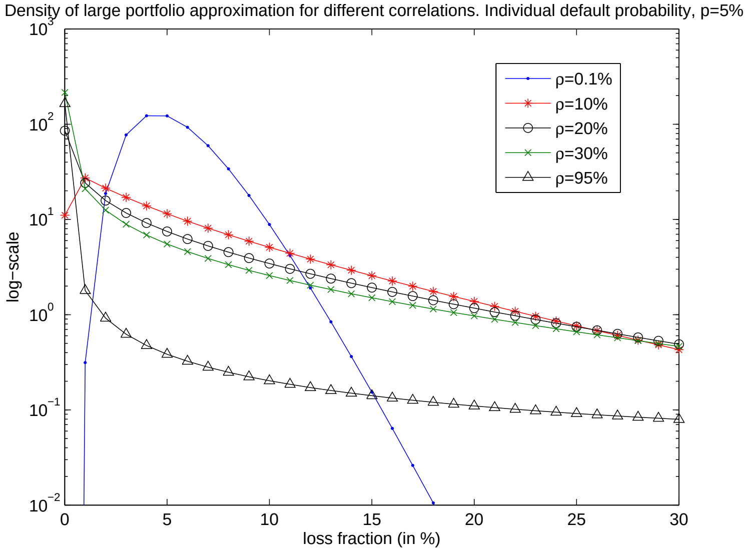
The mixed Merton binomial model and LPA, cont.



The mixed Merton binomial model and LPA, cont.



The mixed Merton binomial model and LPA, cont.



VaR in the mixed binomial Merton model

Consider a static credit portfolio with m obligors in a mixed binomial model inspired by the Merton framework where

- the individual one-year default probability is $\bar{p}$
- the individual loss is ℓ
- the default correlation is ρ

By assuming the LPA setting we can now state the following result for the one-year credit Value-at-Risk $\text{VaR}_\alpha(L)$ with confidence level $1 - \alpha$.

VaR in the mixed binomial Merton model using the LPA setting

With notation and assumptions as above, the one-year $\text{VaR}_\alpha(L)$ is given by

$$\text{VaR}_\alpha(L) = \ell \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right). \tag{32}$$

Useful exercise: Derive the formula (32).

Note that variants of the formula (32) is extensively used for computing regulatory capital in **Basel II** and **Basel III**

Discrete factors in mixed binomial models

- In all our previous examples the random variable Z (modelling the common background factor) have been continuous and the mixing function $p(x) \in [0, 1]$ were chosen to be continuous too.
- However, we can also model Z to be a discrete random variable as follows. Let Z be a random variable such that

$$Z \in \{z_1, z_2, \ldots, z_N\} \quad \text{where} \quad \mathbb{P}[Z = z_n] = q_n \quad \text{and} \quad \sum_{n=1}^{N} q_n = 1. \tag{33}$$

  where it obviously must hold that $q_n \in [0, 1]$ for each $n = 1, 2, \ldots, N$.
- Furthermore, we model the mixing function $p(x) \in [0, 1]$ as

$$p(Z) \in \{p_1, p_2, \ldots, p_N\} \quad \text{where} \quad p(z_n) = p_n \in [0, 1] \quad \text{for each} \quad n \tag{34}$$

  where we without loss of generality may assume that $p_1 < p_2 < \ldots < p_N$.

Discrete factors in mixed binomial models, cont.

- Furthermore, note that

$$\mathbb{P}\left[Z = z_n\right] = \mathbb{P}\left[p(Z) = p_n\right] = q_n \quad \text{for} \quad n = 1, 2, \ldots, N. \tag{35}$$

- Recall that $\bar{p} = \mathbb{P}\left[X_i = 1\right] = \mathbb{E}\left[p(Z)\right]$ so in the model described by (33) and (35) we have

$$\bar{p} = \sum_{n=1}^{N} p_n q_n. \tag{36}$$

- Given (33) and (35) the distribution function $F(x) = \mathbb{P}\left[p(Z) \leq x\right]$ is then for any $x \in [0, 1]$ expressed as

$$F(x) = \sum_{n : p_n \leq x} q_n \,. \tag{37}$$

- Due to the LPA approach we then know that for any $x \in [0, 1]$ it holds that

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \sum_{n : p_n \leq x} q_n \quad \text{as } m \to \infty. \tag{38}$$

Thank you for your attention!