Financial Risk: Credit risk project

Instructions: The following assignment should be handed in before 15.00 on Monday the 22nd of December, 2014. Send the report to `Alexander.Herbertsson@economics.gu.se` and also to the urkund adress `alexander.herbertsson.gu@analys.urkund.se`. If you write the report in word, then *do not convert the word-file into a pdf-file*, unless you use serious converting software. Insert page numbers on each page in your report.

Please read the slides before you start with this assignment. Below all notation are defined as in the slides. Carefully motivate and explain all terms, expressions and computations. Insert your code in in the report, either directly after each task, or in the appendix of the report, with clear references in each task where in the appendix the code can be found. Also send your computer files separately (zipped m-files, excel-files, etc) to `Alexander.Herbertsson@economics.gu.se`. If you send in excel-files, make sure to debug them. The most preferable program is matlab, but other programs are also accepted (however, matlab will indeed be the easiest program for implementing this assignment). You are of course encouraged to contact me when you have questions and make sure to use the scheduled question hours.

Grading:
For grade 3, 4, 5, G and VG, you need to solve the following tasks sufficiently good: 1.1, 1.2, 1.3, 2.1, 2.2
For grade 4, 5 and VG, you also need to solve the following tasks sufficiently good: 2.3, 2.4
For grade 5 and VG, you also need to solve the following tasks sufficiently good: 3.1, 4.1

Good luck!
Alexander

1. The mixed binomial model inspired by the Merton framework

1.1. Consider a static credit portfolio with 1000 obligors and where the size of each loan is 1 million SEK. We model this portfolio as a mixed binomial model inspired by the Merton model with constant default losses given by 60%. Let the individual default probability for each obligor be $\bar{p} = 4\%$ (for a default within one year, say) and assume that the correlation is $\rho = 10\%$. What is the probability that within one year, the total portfolio credit loss will be more than 50 million SEK but less than 100 million SEK. Use the large portfolio approximation formula. Repeat the same computation for the nine pairs $(\bar{p}, \rho)$ given in Table 1 and report the result in a table (hint: in matlab, use the built-in functions `normcdf` and `norminv`).

Table 1. The nine pairs $(\bar{p}, \rho)$. The parameters $\bar{p}$ and ρ are expressed in percent (i.e. $(\bar{p}, \rho) = (4, 25)$ means that $\bar{p} = 0.04$ and $\rho = 0.25$).

$(\bar{p}, \rho) = (4, 15)$	$(\bar{p}, \rho) = (4, 25)$	$(\bar{p}, \rho) = (4, 45)$
$(\bar{p}, \rho) = (8, 15)$	$(\bar{p}, \rho) = (8, 25)$	$(\bar{p}, \rho) = (8, 45)$
$(\bar{p}, \rho) = (16, 15)$	$(\bar{p}, \rho) = (16, 25)$	$(\bar{p}, \rho) = (16, 45)$

1.2. Consider the same portfolio as in the previous task. Use the LPA VaR formula derived in the slides and compute the one-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio where $\alpha = 95\%, 99\%$ and 99.9%. Do this for the nine pairs $(\bar{p}, \rho)$ given in Table 1 and report the result in a table. Hint 1: Use linearity for VaR. Hint 2: compute $\mathrm{ES}_\alpha(L)$ by using e.g. the built-in function `quad` in matlab.

1.3. Consider a homogeneous static credit portfolio which we model as mixed binomial model with mixing probability $p(Z)$ where Z and $p(Z)$ are continuous random variables. Let $F(x) = \mathbb{P}[p(Z) \leq x]$. Assume that the portfolio consists of m obligors where the individual loss is ℓ percent of the notional amount and each loan have notional amount 1. Use the LPA formulas for VaR and ES in order to rigourously prove that

$$1 \leq \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} \leq \frac{1}{F^{-1}(\alpha)} \qquad \text{for all } \alpha \in (0, 1).$$

Verify that the above relation is true for your numerical computations in Task 1.2 and report the quotients $\frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)}$ in tables as well as the values $\frac{1}{F^{-1}(\alpha)}$. Is the theoretical relation in line with your intuition? Discuss your results and findings.

Use the above relation to prove that

$$\lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = 1.$$

Is this relationship in line with your numerical results in Task 1.2 where the computations are done for $\alpha = 95\%, 99\%$ and 99.9%? Is the above limit in line with your intuition? Discuss your results and findings.

2. The mixed binomial logit-normal model

2.1. Consider a homogeneous static credit portfolio which we model as mixed binomial model with a logit-normal mixing distribution. Thus, the mixing probability $p(Z)$ is given by

$$p(Z) = \frac{1}{1 + e^{-(\mu + \sigma Z)}}$$

where $\sigma > 0$ and Z is a standard normal random variable. We assume that $p(Z)$ is, say, the one-year conditional default probability for each obligor. Assume that the portfolio consists of m obligors where the individual loss is ℓ percent of the notional amount and each loan have notional amount 1. Assume that the parameters μ and σ are known and determined for a one-year horizon loss. Use the LPA-approximation formula to derive an analytical expression for the one-year $\mathrm{VaR}_\alpha(L)$ as functions of ℓ, m, σ, μ and α.

2.2. Consider a homogeneous static credit portfolio which we model as mixed binomial model with a logit-normal mixing distribution with mixing probability $p(Z)$ as in the previous task. Compute the one-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio where $\alpha = 95\%, 99\%, 99.9\%$ and do this for the three pairs (μ, σ) given in Table 2. Report the results in a table and discuss your results and findings.

Table 2. The three pairs (μ, σ).

$(\mu, \sigma) = (-3.0588, 0.7202)$	$(\mu, \sigma) = (-2.5371, 0.9899)$	$(\mu, \sigma) = (-2.3961, 1.5508)$

2.3. Consider a homogeneous static credit portfolio which we model as mixed binomial model with a logit-normal mixing distribution with mixing probability $p(Z)$ as in Task 2.1. Compute the individual default probability $\bar{p}_{\mathrm{logN}} = \mathbb{P}[X_i = 1]$ and the pairwise default correlation $\rho_X^{(\mathrm{logN})}$ for the three pairs (μ, σ) given by Table 2 (recall that $\rho_X^{(\mathrm{logN})} = \mathrm{Corr}(X_i, X_j)$ for $i \neq j$ where $\mathbb{P}[X_i = 1 \,|\, Z] = p(Z)$ and $p(Z)$ is given as in Task 2.1). Report the values of $\bar{p}_{\mathrm{logN}}$ and $\rho_X^{(\mathrm{logN})}$ in a table. Discuss your results and findings. Are they in line with the corresponding VaR and ES values computed in the previous task.

2.4. Consider the same portfolio and model as in the previous task. For each pair (μ, σ) in Table 2, find the corresponding parameters $(\bar{p}, \rho)$ in a mixed binomial Merton model so that $\bar{p} = \bar{p}_{\mathrm{logN}}$ and $\rho_X^{(\mathrm{logN})} = \rho_X^{(M)}$ where $\rho_X^{(M)}$ is the pairwise default correlation in the mixed binomial Merton model. Report the values $(\bar{p}, \rho)$ in a table. Compute the one-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio in a mixed binomial Merton model where $\alpha = 95\%, 99\%$ and 99.9% for these three parameters $(\bar{p}, \rho)$ and compare them with the corresponding VaR and ES values in Task 2.2. This means that we compare VaR and ES in two different models (mixed binomal logit Normal vs mixed binomial Merton) with same individual default probability and same pairwise default correlations in both models (for all three different parameter cases). Hence, the comparison between the two models is somewhat "fair"/relevant and could be one way of investigating so called model risk. Discuss your results and findings, for example, does VaR and ES differ much in the two different models etc. Hint: To find ρ use `fsolve`.

3. The mixed beta binomial model

3.1. Consider a static credit portfolio with m obligors which we model as mixed beta binomial model with parameter a and b. The individual loss ℓ is 60% of the notional amount which is the same for all obligors and given by 1. For $m = 50$, compute $\mathrm{VaR}_\alpha(L)$ where $\alpha = 95\%, 99\%$ and 99.9%, both with the exact method (i.e. using a generalized inverse) and with the LPA formula. Also compute the individual default probability $\bar{p}$ and the pairwise default correlation $\rho_X^{(B)}$ (recall that $\rho_X^{(B)} = \mathrm{Corr}(X_i, X_j)$ for $i \neq j$ where $\mathbb{P}[X_i = 1 \,|\, Z] = p(Z) = Z$ is beta-distributed with parameter a and b). Do this for the parameters $(a, b) = (\lambda, 9\lambda)$ where $\lambda = 0.4, 2, 4, 20$. Report the results in a table and discuss your results and findings, for example is the LPA-VaR formula a good approximation to the exact VaR formula. (hint: in matlab, use the built-in functions `beta` and `betainv`. Use a `while`-loop to find the generalized inverse).

4. Random losses in the mixed binomial Merton model

4.1. Consider a static homogeneous credit portfolio with 1000 obligors and where the size of each loan is 1 monetary unit. We model this portfolio as a mixed binomial model inspired by the Merton model where the individual default probability for each obligor is $\bar{p}$ (for a default within one year, say) and the correlation is ρ. Let Z be the mixing variable in the mixed binomial Merton model, that is Z is a standard normal, and let $p(Z)$ be the conditional one-year default probability, given by the same formula as in the slides, that is

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right).$$

Next, we assume that the individual losses $\ell_1 = \ell_1(Z), \ell_2 = \ell_2(Z), \ldots, \ell_m = \ell_m(Z)$ are random such that conditionally on Z then $\ell_1(Z), \ell_2(Z), \ldots, \ell_m(Z)$ are i.i.d with distribution given by $\ell(Z)$ such that

$$\ell(Z) = \begin{cases} 0.5 & \text{if } p(Z) \leq c \cdot \bar{p} \\ 0.9 & \text{if } p(Z) > c \cdot \bar{p} \end{cases}$$

where $c = 0.5$. Give an economic/financial interpretation of the above distribution $\ell(Z)$. Use the LPA formula to derive an explicit expression for the portfolio loss distribution with random individual losses as above. Next, for the nine pairs $(\bar{p}, \rho)$ given in Table 1, compute the LPA-VaR values $\mathrm{VaR}_\alpha(L)$ for $\alpha = 95\%, 99\%$ and 99.9%, both with stochastic individual losses as specified above, and with constant losses given by $\bar{\ell} = \mathbb{E}[\ell(Z)]$. Note that $\bar{\ell}$ will vary with $(\bar{p}, \rho)$. Report the results in a table and discuss your results and findings, for example are there big differences between VaR-values when computed with a constant loss $\bar{\ell}$ versus a stochastic loss $\ell(Z)$ where $\bar{\ell} = \mathbb{E}[\ell(Z)]$. How sensitive are the VaR-values when c changes to some other constant. Hint: To find $\mathrm{VaR}_\alpha(L)$ in the stochastic loss case, use `fsolve`.