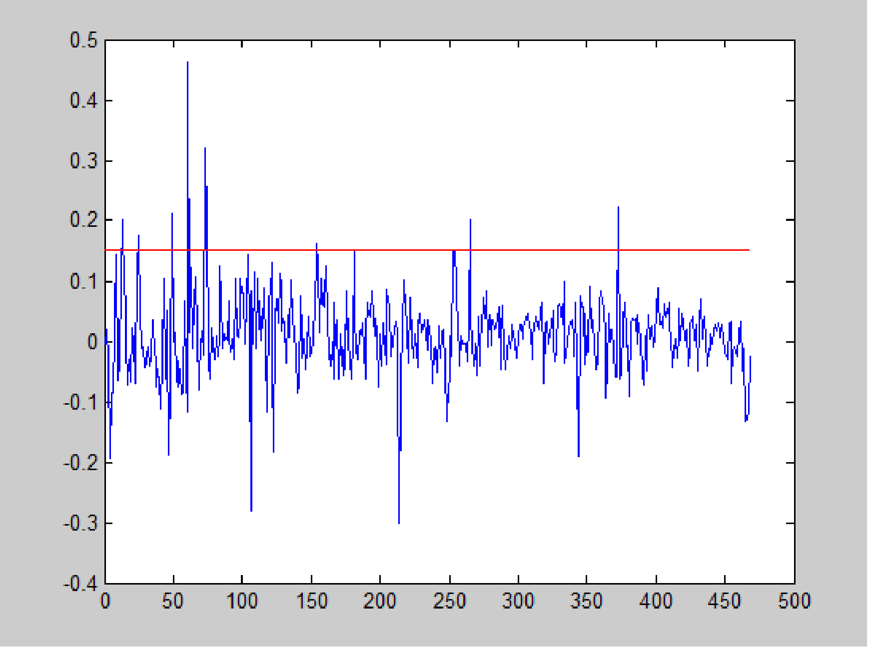
Financial Risk

Lecture 3

Peaks over Threshold (PoT)

- Instead of using block maxima to model behaviour of big losses (or big wins) we may use observations above certain level or threshold



Peaks over Threshold (PoT)

- Letting t_i denote a time where an exceedance (assumed to arrive according to a Poisson process) of the threshold η occurs we will use the data

$$\{r_{t_1} - \eta, \dots, r_{t_n} - \eta\}$$

- Or, if we wish to model losses, rather

$$\{-r_{t_1} - \eta, \dots, -r_{t_n} - \eta\} \text{ or } \{z_{t_1} - \eta, \dots, z_{t_n} - \eta\}$$

- In PoT we do not have to choose a block size but a threshold η and different choices will give different estimates but it turns out that VaR done with PoT may not be as sensitive to the threshold choice as VaR done with block maxima is to the choice of block length

- Starting out with a value of η so that 5% of the sample is left is a rule of thumb

Which distribution?

- Since we will just use returns (negated or standardized negated, that is) above η the distribution we will use is conditional on the event $r_t > \eta$

$$P(r_t \le x + \eta \,|r_t > \eta\,) = \frac{P(r_t \le x + \eta) - P(r_t \le \eta)}{1 - P(r_t \le \eta)}$$

- Using $P(r_t \le x) = exp\left[-\left(1 + \xi\,\frac{x-\beta}{\alpha}\right)^{-1/\xi}\right]$ and the approximation $e^{-x} \approx 1 - x$ gives (exercise)

$$P(r_t \le x + \eta \,|r_t > \eta\,) \approx 1 - \left[1 + \frac{\xi x}{\alpha + \xi(\eta - \beta)}\right]^{-1/\xi}$$

where $x > 0$ and $\alpha + \xi(\eta - \beta) > 0$

GDP

- We refer to the above distribution as the Generalized Pareto Distribution

- The limiting case $\xi = 0$ gives

$$P(r_t \leq x + \eta \,|r_t > \eta\,) \approx 1 - exp[-x/\alpha]$$

- We sometimes write (the scale parameter)

$$\psi(\eta) = \alpha + \xi(\eta - \beta)$$

Important properties

- If a certain threshold η_0 yields a shape parameter ξ and scale parameter $\psi(\eta_0)$, a higher threshold $\eta > \eta_0$ will yield the scale parameter

$$\psi(\eta) = \psi(\eta_0) + \xi(\eta - \eta_0)$$

- For the case $\xi = 0$ the GPD is just an exponential distribution so peaks over threshold should in this case behave according to an exponential distribution

Mean Excess Function

- Another tool for choosing threshold is using the mean excess over the threshold η_0, assuming $\xi < 1$ (exercise)

$$E(r_t - \eta_0 | r_t > \eta_0) = \frac{\psi(\eta_0)}{1 - \xi}$$

- Then for any $\eta > \eta_0$ we have the mean excess function

$$e(\eta) = E(r_t - \eta | r_t > \eta) = \frac{\psi(\eta_0) + \xi(\eta - \eta_0)}{1 - \xi}$$

- So for a fixed ξ the mean excess function is linear in $\eta - \eta_0$

Empirical Mean Excess Function

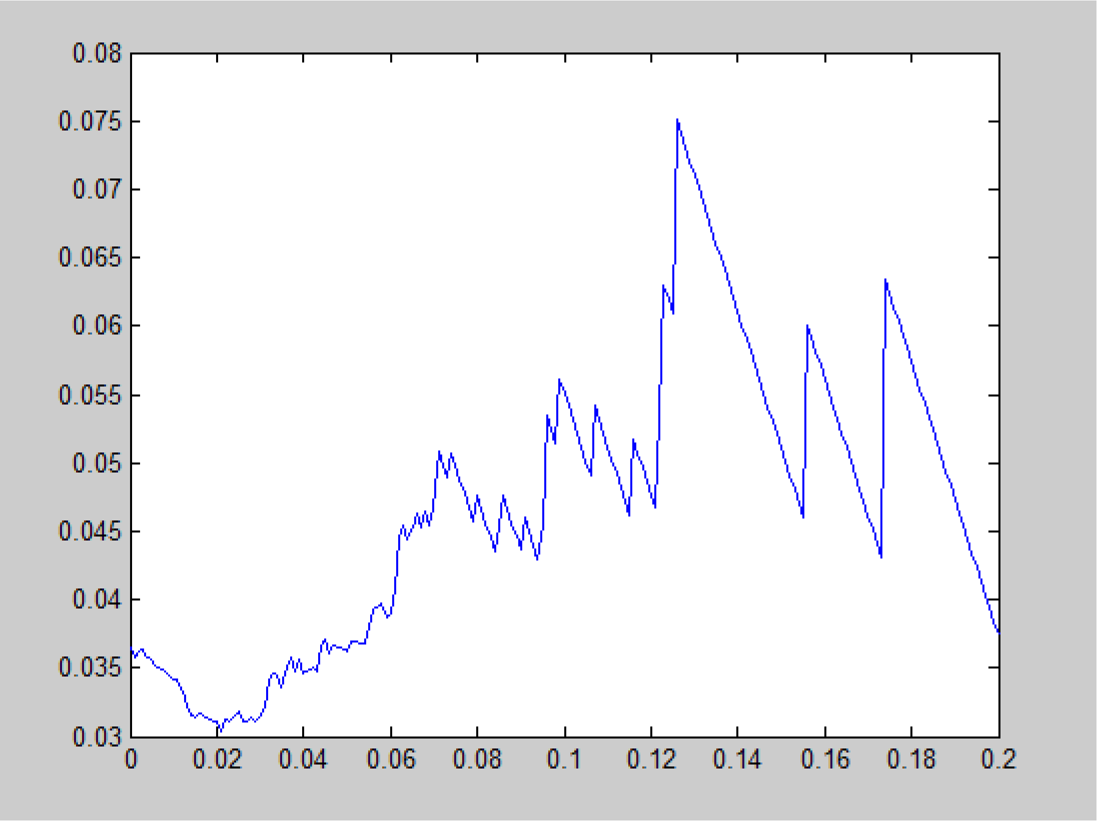
- In practice we will use the empirical mean excess function

$$e_T(\eta) = \frac{1}{N(\eta)} \sum_{i=1}^{N(\eta)} \left(r_{t_i} - \eta\right)$$

where $N(\eta)$ is the number of exceedances of η

- We plot $e_T(\eta)$ against η and choose the threshold η_0 as the value for which $\eta > \eta_0$ gives a linear appearance of $e_T(\eta)$

For negated decile 9 data



Decile 9

- We are working with negated returns since we are interested in risks of big losses

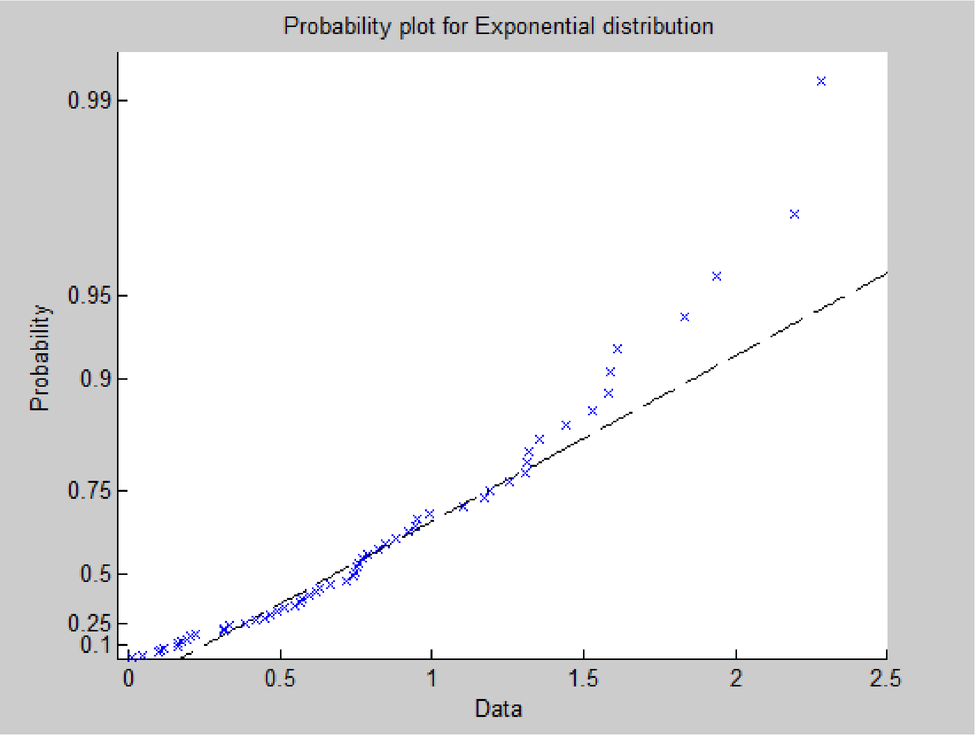
- Based on the the mean excess plot we choose $\eta_0 = 0.04$ and use ML (Matlab or R) to fit the GPD to the series $\{r_{t_i} - 0.04\}$ for those r_{t_i} that exceed the threshold

- The parameter estimates are
$$\hat{\alpha} = -0.0275, \hat{\beta} = 0.0029, \hat{\xi} = 1.2049$$

- Note that the validity of the mean excess plot may be questioned as $\hat{\xi} = 1.2049 > 1$

Model checking

- We define residuals as $\frac{1}{\hat{\xi}} \ln \left(1 + \hat{\xi}\, \frac{r_{t_i} - \eta_0}{\hat{\alpha} + \hat{\xi}(\eta_0 - \hat{\beta})}\right)$ which should follow an exponential distribution



Model checking

- The above probability plot was made using the estimated parameters using the threshold 0.04 suggested by the mean excess plot

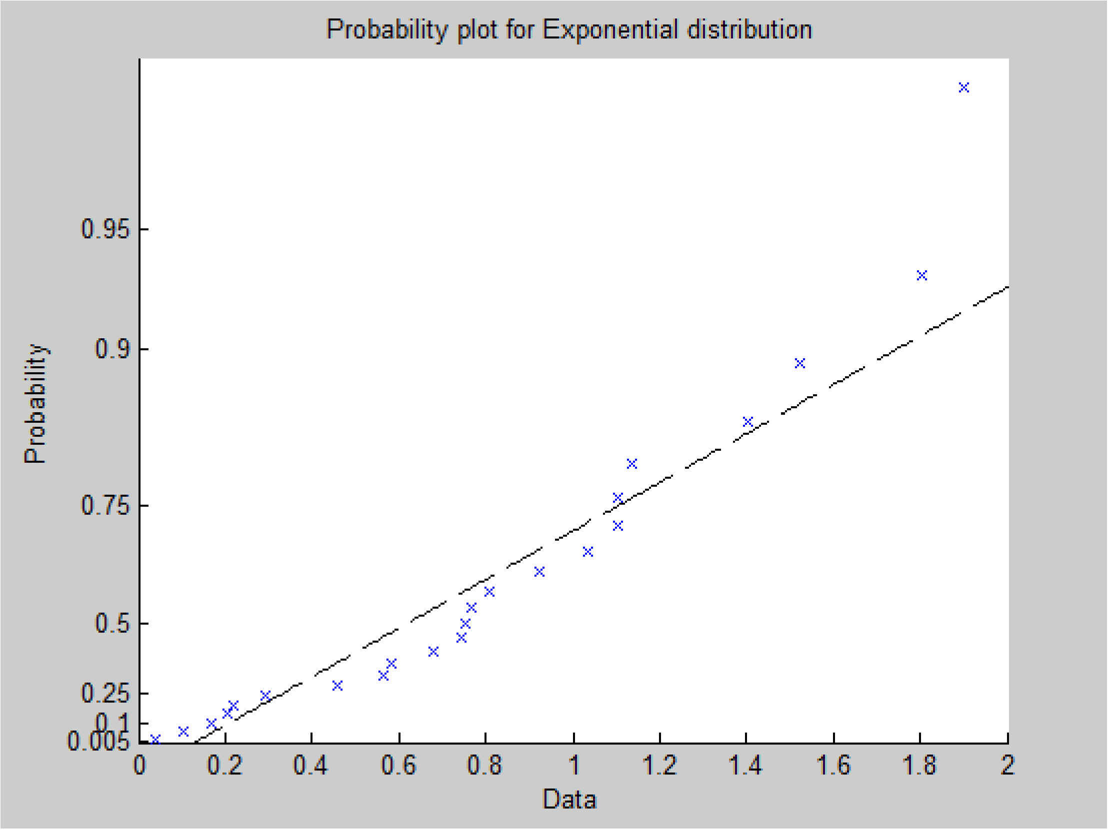
- However the residual plot suggests trying a larger threshold (to be able to capture the big losses) and we will use the 5% rule of thumb which yields $\eta_0 = 0.0644$

- Re-estimation gives

$$\hat{\alpha} = -0.0492, \hat{\beta} = 0.0030, \hat{\xi} = 1.2072$$

Note: The above (three) estimates were made using a ”hand coded” likelihood function in matlab. Using ”gpfit” in matlab will give (two) estimates of ξ and $\psi(\eta) = \alpha + \xi(\eta - \beta)$ when plugging in $\{-r_{t_i} - \eta\}$

Model checking



QQ-plots

- Inversion of the distribution function

$$y = \begin{cases} 1 - \left[1 + \dfrac{\xi x}{\alpha + \xi(\eta - \beta)}\right]^{-1/\xi} \\ 1 - \exp\left\{-\dfrac{x}{\alpha}\right\} \end{cases}$$

yields (verify this)

$$x = \begin{cases} \dfrac{\alpha + \xi(\eta - \beta)}{\xi}\left((1 - y)^{-\xi} - 1\right) \\ -\alpha \ln(1 - y) \end{cases}$$

QQ-plots

- So if we denote the ordered (in increasing size) exceedances by $r_{(i)} - \eta$, and let $\widehat{\psi} = \hat{\alpha} + \hat{\xi}(\eta - \hat{\beta})$, for $i = 1, \dots, k$, the pairs

$$\left(\frac{\widehat{\psi}}{\hat{\xi}}\left(\left(\frac{k+1-i}{k+1}\right)^{-\hat{\xi}} - 1\right), r_{(i)} - \eta\right)$$

or

$$\left(-\hat{\alpha}\ln\left(\frac{k+1-i}{k+1}\right), r_{(i)} - \eta\right)$$

should form a straight line

- Note that r (return) is replaced by z (standardized negated) when using the dynamic scheme discussed in lecture 1

VaR using PoT

- So we know how to estimate $P(r_t \leq x + \eta \,|r_t > \eta\,)$ since, letting $y = x + \eta$,

$$P(r_t \leq y|r_t > \eta\,) = \frac{F(y) - F(\eta)}{1 - F(\eta)} \approx \begin{cases} 1 - \left[1 + \dfrac{\xi(y-\eta)}{\alpha + \xi(\eta - \beta)}\right]^{-1/\xi} \\ 1 - \exp\left\{-\dfrac{y-\eta}{\alpha}\right\} \end{cases}$$

- This gives (why?), where T is the total number of observations in the returns series

$$F(y) \approx \begin{cases} 1 - \dfrac{N(\eta)}{T}\left[1 + \dfrac{\xi(y-\eta)}{\alpha + \xi(\eta - \beta)}\right]^{-1/\xi} \\ 1 - \dfrac{N(\eta)}{T}\exp\left\{-\dfrac{y-\eta}{\alpha}\right\} \end{cases}$$

VaR using PoT

- So for a small upper tail probability q, letting $p = 1 - q$ we have

$$VaR_p = \begin{cases} \eta - \dfrac{\alpha + \xi(\eta - \beta)}{\xi}\left\{1 - \left[\dfrac{T}{N(\eta)}(1-p)\right]^{-\xi}\right\} \\ \eta - \alpha \ln\left(\dfrac{T}{N(\eta)}(1-p)\right) \end{cases}$$

- Also the associated expected shortfall, at least for $\xi < 1$, is given by

$$ES_p = \frac{VaR_p + \alpha - \xi\beta}{1 - \xi}$$

Decile 9

- Using the parameter estimates above and that we have 23 exceedances in 468 observations we get we get the 95% and 99% VaR for the decile 9 data; 0.0640 and 0.1849

- Compare to 0.0592 and 0.1091 from Block-Maxima method

Simple scheme available

- It is hard, if not impossible, to fit GP distributions using Excel, without using some add-ons on plug-ins unless…

- If one assumes that $\xi = 0$ it holds that the expected value is α that the standard deviation is α so we may estimate α using

$$\hat{\alpha} = \bar{X} \text{ or } \hat{\alpha} = S$$

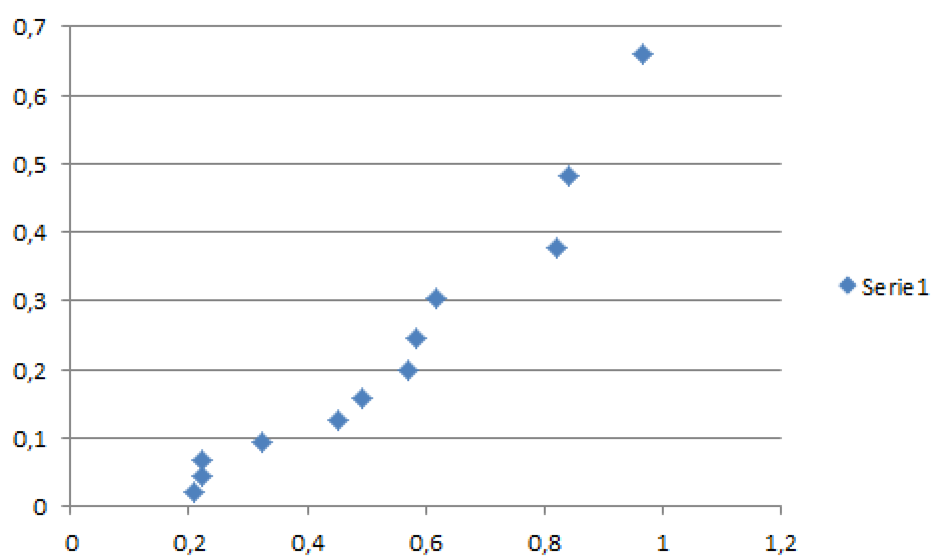
where $\bar{X}$ and S are the mean and standard deviation of observations (negated returns minus threshold for negated returns over the threshold)

- So, if we do not assume $\xi = 0$ we probably will have to use Matlab or R or some other package that we like

- In turns out that using this simple scheme for Facebook data seems to underestimate VaR.

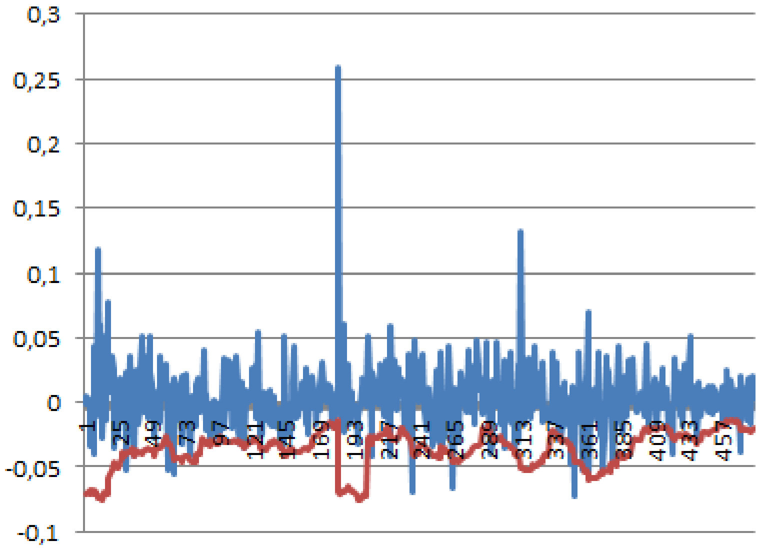
Simple scheme for FB

- Some tweaking gives an ok fit for peaks over the threshold 1.7 (QQ-plot made in Excel)



Simple scheme available

- Chosing 1.7 as threshold for the standardized negated FB returns gives a 95%-quantile of 1.50 to be used for VaR
- This gives 6% violations and hence VaR is slightly under-estimated
- That this method under-estimates the risk for the data at hand does not necessarily mean that it under-estimates risks for all data…



Relation between BM and PoT

- Assuming that exceedances of the threshold η arrive according to a Poisson process with intensity λ and that sizes of exceedances are i.i.d. GP-distributed and independent of the PP, we have, where $H(x)$ denotes the GP distribution function and where $\psi(\eta) = \alpha + \xi(\eta - \beta)$, for the maximum M_T over the time interval $[0, T]$ that M_T is GEV distributed;

Relation between BM and PoT

$$P(M_T \leq \eta + x) = \sum_{k=0}^{\infty} P(M_T \leq \eta + x, k \text{ exceedances in } [0,T])$$

$$= \sum_{k=0}^{\infty} \big(H(x)\big)^k \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$= \sum_{k=0}^{\infty} \left(1 - \left[1 + \frac{\xi x}{\psi(\eta)}\right]^{-1/\xi}\right)^k \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$= \exp\left\{-\left(1 + \xi \frac{x - \psi(\eta)\big((\lambda T)^\xi - 1\big)/\xi}{\psi(\eta)(\lambda T)^\xi}\right)^{-1/\xi}\right\}$$

The last expression is that of a GEV distribution function!

Summary and about tech proj 1

- We have seen that it is possible to compute and forecast GP (with shape parameter zero) VaR using simple methods
- What has been covered today will be included in technical project 1 and in particular one must know how to create a dynamic GP (with shape parameter zero) VaR series under i.i.d. assumption starting from an arbitrary set of stock prices in order to get a grade higher than 3/G
- Hence it is recommended that you download some data set, preferrably two years of prices, from e.g. yahoo finance and starting playing around with it.
- Note that file downloaded from yahoo finance are csv-files that you may transfer to xlsx by using ”text to columns” under ”data” in Excel
- To get started check out the xlsx-file available on facebook and at the course web page