Financial Risk VT 2017: Credit risk project

Instructions: The following assignment should be handed in before 17.00 on Thursday May 18th, 2017. Send the report to
`alexander.herbertsson@cff.gu.se` or `alexander.herbertsson@economics.gu.se`
and make a cc to the urkund adress `alexander.herbertsson.gu@analys.urkund.se`
If you do not send it to the urkund adress, the project will not pass. If you write the report in word, then *do not convert the word-file into a pdf-file*, unless you use serious converting software. Insert page numbers on each page in your report.

Before you start with the project, please carefully read the "Student ethics" document on the webpage of the course, or on the link
`www.math.chalmers.se/Stat/Grundutb/CTH/mve220/1617/ethics.pdf`
and make sure that you understand the message in this document.

Please carefully read the corresponding parts in the lecture notes and the slides before you start with this assignment. Below all notation are defined as in the lecture notes and the slides. Carefully motivate and explain all terms, expressions and computations in your report. The most preferable program is matlab which also is the easiest tool for implementing this assignment. Insert your m-code in the report, either directly after each task, or in the appendix of the report, with clear references in each task where in the appendix the code can be found. Also send your computer files separately (zipped m-files) to `alexander.herbertsson@cff.gu.se` or `alexander.herbertsson@economics.gu.se`
You are of course encouraged to contact me when you have concrete questions, but first carefully read the lecture notes and the slides to see if your question have an immediate answer. You can also ask your questions regarding the project after the lectures.

Grading: In order to pass this project all tasks must be solved sufficiently good

Good luck!
Alexander

1. The mixed binomial model inspired by the Merton framework

1.1. Consider a static credit portfolio with 1000 obligors and where the size of each loan is 1 million SEK. We model this portfolio as a mixed binomial model inspired by the Merton model with constant default losses given by 60%. Let the individual default probability for each obligor be $\bar{p} = 4\%$ (for a default within one year, say) and assume that the correlation is $\rho = 10\%$. What is the probability that within one year, the total portfolio credit loss will be more than 30 million SEK but less than 80 million SEK. Use the large portfolio approximation formula. Repeat the same computation for the nine pairs $(\bar{p}, \rho)$ given in Table 1 and report the result in a table (hint: in matlab, use e.g. the built-in functions `normcdf` and `norminv`).

Table 1. The nine pairs $(\bar{p},\rho)$. The parameters $\bar{p}$ and ρ are expressed in percent (i.e. $(\bar{p},\rho) = (5,15)$ means that $\bar{p} = 0.05$ and $\rho = 0.15$).

$(\bar{p},\rho) = (4,15)$	$(\bar{p},\rho) = (4,40)$	$(\bar{p},\rho) = (4,65)$
$(\bar{p},\rho) = (8,15)$	$(\bar{p},\rho) = (8,40)$	$(\bar{p},\rho) = (8,65)$
$(\bar{p},\rho) = (12,15)$	$(\bar{p},\rho) = (12,40)$	$(\bar{p},\rho) = (12,65)$

1.2. Consider a static credit portfolio with m obligors which we model as mixed binomial model inspired by the Merton framework. The individual one-year default probability is $\bar{p}$, the individual loss is ℓ, and the default correlation is ρ. Use the large portfolio approximation (LPA) formula to derive an analytical expression for the one-year $\mathrm{VaR}_\alpha(L)$ as functions of $\ell, m, \rho, \bar{p}$ and α.

1.3. Consider the same portfolio as in Task 1.1 above. Use the LPA-VaR formula derived in Task 1.2 and compute the one-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio where $\alpha = 95\%, 99\%$ and 99.9%. Do this for the nine pairs $(\bar{p},\rho)$ given in Table 1 and report the result in a table. Hint 1: Use linearity for VaR. Hint 2: compute $\mathrm{ES}_\alpha(L)$ by using e.g. the built-in function `integral` in matlab.

2. The mixed beta binomial model

2.1. Consider a static credit portfolio with m obligors which we model as mixed beta binomial model with parameter a and b for one year, that is the one-year conditional default probability $p(Z)$ is given by $p(Z) = \mathbb{P}[X_i = 1 \mid Z] = Z$ where $Z \sim \mathrm{Beta}(a,b)$. Derive an expression for the individual (one-year) default probability $\bar{p}$ and the pairwise default correlation $\rho_X^{(B)} = \mathrm{Corr}(X_i, X_j)$ for $i \neq j$ as functions of the parameters a and b. Conversely, find an expression of a and b in terms of $\bar{p}$ and $\rho_X^{(B)}$. Carefully motivate and explain your computations; it is not enough to just state the formulas without a derivation.

2.2. Consider a static credit portfolio with m obligors which we model as mixed beta binomial for one year, as in Task 2.1. Let $m = 1000$ and the loss be 60% of the individual notational where each loan have notational 1 million SEK. Given the values for the pair $(\bar{p}, \rho_X^{(B)})$ in Table 2, use the LPA-VaR formula to compute the 1-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio where $\alpha = 95\%, 99\%$ and 99.9%. Do this for the six columns in Table 2 and report the result in tables. Discuss your results and findings, for example how is the VaR-values related to the values $\bar{p}$ and $\rho_X^{(B)}$ in Table 2. Hint 1: Use the results from Task 2.1. Hint 2: compute $\mathrm{ES}_\alpha(L)$ by using e.g. the built-in function `integral` in matlab. Also, in matlab, use the built-in function `betainv`.

Table 2. The six values of the pair $(\bar{p}, \rho_X^{(B)})$.

$\bar{p}$	0.04	0.04	0.1	0.1	0.15	0.15
$\rho_X^{(B)}$	0.15	0.75	0.15	0.75	0.15	0.75

3. The mixed binomial logit-normal model

3.1. Consider a static credit portfolio which we model as mixed binomial model with a logit-normal mixing distribution. Thus, the one-year conditional default probability $p(Z)$ is given by

$$p(Z) = \frac{1}{1 + e^{-(\mu+\sigma Z)}}$$

where $\sigma > 0$ and Z is a standard normal random variable.

In order to use this model in practice we need to determine the parameters σ and μ. This can be done in many ways. Here we choose to estimate σ and μ by calibrating our mixed binomial logit-normal model against a mixed binomial model inspired by the Merton framework, by letting the LPA-distributions agree at two different points. Let $F_M(x)$ and $F_{\mathrm{logN}}(x)$ be the LPA-distributions for the fractional number of defaults in the portfolio, for a mixed binomial Merton model and for the mixed binomial logit-normal model. Let $x_1, x_2 \in (0, 1)$ where $x_1 < x_2$ and assume that the Merton parameters $\bar{p}, \rho$ are known. We then calibrate σ and μ so that

$$F_M(x_i) = F_{\mathrm{logN}}(x_i) \quad \text{for } i = 1, 2. \tag{3.1}$$

Intuitively this means that we determine σ and μ so that the probability of having a fractional number of defaults less or equal to $100 \cdot x_i$ percent is the same in both models for the two fixed values x_1 and x_2. Use Equation (3.1) to find σ and μ as explicit functions of x_1, x_2 and the Merton parameters $\bar{p}$ and ρ, that is, derive an explicit expression of σ and μ as functions of the parameters $x_1, x_2, \bar{p}, \rho$.

Hint: consider $x_1, x_2, \bar{p}, \rho$ as fixed constants and solve a system of linear equations for σ and μ. For notational convenience, let $a(x, \bar{p}, \rho) = \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p})\right)$ in order to simply the obtained expressions for σ and μ.

3.2. Consider a static credit portfolio with m obligors which we model as mixed binomial logit-normal model as in task 3.1. Assume that the parameters μ and σ are known and determined for a one-year horizon loss. Use the LPA-approximation formula to derive an analytical expression for the one-year $\mathrm{VaR}_\alpha(L)$ as functions of ℓ, m, σ, μ and α.

3.3. Consider a static credit portfolio with 1000 obligors. We model this portfolio as a mixed binomial logit-normal model as in task 3.1, where the individual loss is 60% of the notational and each loan have notational of 1 million SEK. This logit-model is calibrated against the mixed binomial Merton model for $x_1 = 0.1$ and $x_2 = 0.9$ as described in task 3.1 where the default probability is for one year. In the calibrated logit-model, use the formula derived in task 3.2 and compute the 1-year $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for this portfolio where $\alpha = 95\%, 99\%$ and 99.9%. Do this for the nine pairs $(\bar{p}, \rho)$ given in Table 1, where $(\bar{p}, \rho)$ is used in the Merton model which is calibrated against the logit-normal model.

Report the results in a table and compare with the results in task 1.3. Discuss your results and findings.