Financial Risk: Credit Risk, Lecture 1

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April 25, 2017

Main goal of lectures and content of today's lecture

- The main goal of coming three lectures is to study the loss distribution for a credit portfolio
- The loss distribution is used to compute risk measures such as Value-at-Risk etc.

Today's lecture

- Short discussion of the important components of credit risk
- Study different static portfolio credit risk models.
- Discussion of the binomial loss model
- Discussion of the mixed binomial loss model

Definition of Credit Risk

Credit risk

— the risk that an obligor does not honor his payments

Example of an obligor:

- A company that have borrowed money from a bank
- A company that has issued bonds.
- A household that have borrowed money from a bank, to buy a house
- A bank that has entered into a bilateral financial contract (e.g an interest rate swap) with another bank.

Example of defaults are

- A company goes bankrupt.
- As company fails to pay a coupon on time, for some of its issued bonds.
- A household fails to pay amortization or interest rate on their loan.

Credit Risk

Credit risk can be decomposed into:

- **arrival risk**, the risk connected to whether or not a default will happen in a given time-period, for a obligor
- **timing risk**, the risk connected to the uncertainness of the exact time-point of the arrival risk (will not be studied in this course)
- **recovery risk**. This is the risk connected to the size of the actual loss if default occurs (will not be studied in this course, we let the recovery be fixed)
- **default dependency risk**, the risk that several obligors jointly defaults during some specific time period. This is one of the most crucial risk factors that has to be considered in a credit portfolio framework.
- The coming three lectures focuses only on **default dependency risk**.

Portfolio Credit Risk is important

- Portfolio credit risk models differ greatly depending on what types of portfolios, and what type of questions that should be considered. For example,
  - models with respect to risk management, such as credit Value-at-Risk (VaR) and expected shortfall (ES)
  - models with respect to valuation of portfolio credit derivatives, such as CDO´s and basket default swaps
- In both cases we need to consider **default dependency risk**, but....
- ...in risk management modelling (e.g. VaR, ES), the timing risk is ignored, and one often talk about **static credit portfolio models**,
- ...while, when pricing credit derivatives, timing risk must be carefully modeled (not treated here)
- The coming three lectures focuses only on **static credit portfolio models**,

Literature

The slides for the coming three lectures are rather self-contained, but more details on certain topics can be found in the lecture notes.

The content of the lecture today and the next lecture is **partly** based on materials presented in:

- *"Quantitative Risk Management"* by McNeil A., Frey, R. and Embrechts, P. (Princeton University Press)
- *"Credit Risk Modeling: Theory and Applications"* by Lando, D . (Princeton University Press)
- *"Risk and portfolio analysis - principles and methods"* by Hult, Lindskog, Hammerlid and Rehn. (Springer)

Static Models for homogeneous credit portfolios

- Today we will consider the following static modes for a homogeneous credit portfolio:
  - The binomial model
  - The mixed binomial model
- To understand mixed binomial models, we give a short introduction of conditional expectations
- In the next two lectures we will
  - study three different mixed binomial models.
  - discuss Value-at-Risk and Expected shortfall in a mixed binomial models.
  - Correlations etc in mixed binomial models.

The binomial model for independent defaults

Consider a homogeneous credit portfolio model with m obligors where each obligor can default up to fixed time T, and have the same constant credit loss ℓ.

- Let X_i be a random variable such that

$$X_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults before time } T \\ 0 & \text{otherwise, i.e. if obligor } i \text{ survives up to time } T \end{cases} \tag{1}$$

- We assume that $X_1, X_2, \ldots X_m$ are **i.i.d**, that is they are **i**ndependent with **i**dentical **d**istribution. Furthermore $\mathbb{P}[X_i = 1] = p$ so $\mathbb{P}[X_i = 0] = 1 - p$.

- The total credit loss in the portfolio at time T, called L_m, is then given by

$$L_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

  thus, N_m is the **number** of defaults in the portfolio up to time T.

- Since ℓ is a constant, we have $\mathbb{P}[L_m = k\ell] = \mathbb{P}[N_m = k]$, so it is enough to study the distribution of N_m.

The binomial model for independent defaults, cont.

- Since $X_1, X_2, \dots X_m$ are i.i.d with $\mathbb{P}[X_i = 1] = p$ we see that $N_m = \sum_{i=1}^m X_i$ is binomially distributed with parameters m and p, i.e. $N_m \sim Bin(m, p)$.

- Hence, we have

$$\mathbb{P}[N_m = k] = \binom{m}{k} p^k (1-p)^{m-k}$$

- Recalling the binomial theorem $(a+b)^m = \sum_{k=0}^m \binom{m}{k} a^k b^{m-k}$ we see that

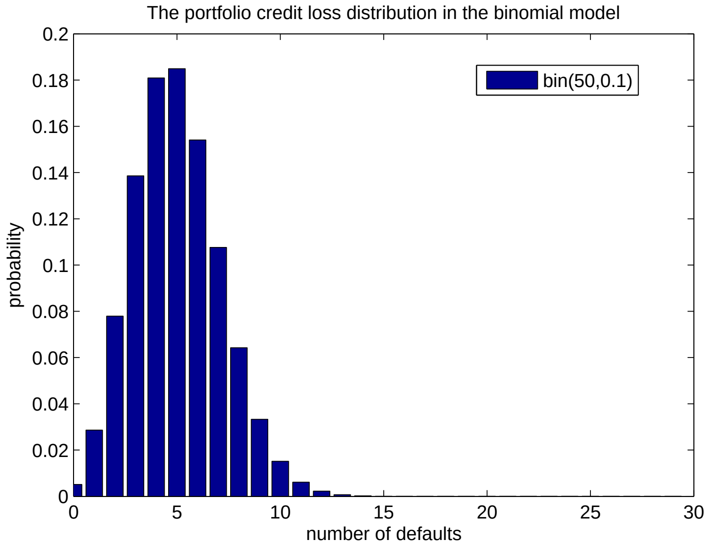
$$\sum_{k=0}^m \mathbb{P}[N_m = k] = \sum_{k=0}^m \binom{m}{k} p^k (1-p)^{m-k} = (p + (1-p))^m = 1$$

  proving that $Bin(m, p)$ is a distribution.

- Furthermore, $\mathbb{E}[N_m] = mp$ since

$$\mathbb{E}[N_m] = \mathbb{E}\left[\sum_{i=1}^m X_i\right] = \sum_{i=1}^m \mathbb{E}[X_i] = mp.$$

The binomial model for independent defaults, cont.



The binomial model for independent defaults, cont.

- The binomial distribution have very thin "tails", that is, it is extremely unlikely to have many losses (see figure).
- For example, if $p = 5\%$ and $m = 50$ we have that $\mathbb{P}[N_m \geq 8] = 1.2\%$ and for $p = 10\%$ and $m = 50$ we get $\mathbb{P}[N_m \geq 10] = 5.5\%$
- The main reason for these small numbers is due to the independence assumption for $X_1, X_2, \ldots X_m$.
- To see this, recall that the variance $\mathrm{Var}(X)$ measures the degree of the deviation of X around its mean, i.e. $\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$.
- Since $X_1, X_2, \ldots X_m$ are independent we have that

$$\mathrm{Var}(N_m) = \mathrm{Var}\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} \mathrm{Var}(X_i) = mp(1-p) \tag{2}$$

where the second equality is due the independence assumption.

The binomial model for independent defaults, cont.

- Furthermore, by Chebyshev's inequality we have that for any random variable X, and any $c > 0$ it holds

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq c\right] \leq \frac{\text{Var}(X)}{c^2}.$$

- Example: if $p = 5\%$ and $m = 50$ then $\text{Var}(N_m) = 50p(1-p) = 2.375$ and $\mathbb{E}[N_m] = 50p = 2.5$.

- So with $p = 5\%$ and $m = 50$, the probability of say, 6 more or less losses than expected, is smaller or equal than 6.6%, since by Chebyshev

$$\mathbb{P}\left[|N_m - 2.5| \geq 6\right] \leq \frac{2.375}{36} = 6.6\%.$$

- Next we show that the deviation of the fractional number of defaults in the portfolio, $\frac{N_m}{m}$, from the constant $p = \mathbb{E}\left[\frac{N_m}{m}\right]$, goes to zero as $m \to \infty$.

- So $\frac{N_m}{m}$ converges towards a constant as $m \to \infty$ (the law of large numbers).

Independent defaults and the law of large numbers

- By applying Chebyshev's inclusion inequality to $\frac{N_m}{m}$ together with Equation (2) we get

$$\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq \varepsilon\right] \leq \frac{\mathrm{Var}\left(\frac{N_m}{m}\right)}{\varepsilon^2} = \frac{\frac{1}{m^2}\mathrm{Var}\left(N_m\right)}{\varepsilon^2} = \frac{mp(1-p)}{m^2\varepsilon^2} = \frac{p(1-p)}{m\varepsilon^2}$$

- Thus, $\mathbb{P}\left[\left|\frac{N_m}{m} - p\right| \geq \varepsilon\right] \to 0$ as $m \to \infty$ for any $\varepsilon > 0$.
- This result is called the weak law of large numbers
- For our credit portfolio it means that the fractional number of defaults in the portfolio, i.e. $\frac{N_m}{m}$, converges (in probability) to the constant p, i.e the individual default probability.
- One can also show the so called strong law of large numbers, that is

$$\mathbb{P}\left[\frac{N_m}{m} \to p \;\text{ when }\; m \to \infty\right] = 1$$

and we say that $\frac{N_m}{m}$ converges almost surely to the constant p. In these lectures we write $\frac{N_m}{m} \to p$ to indicate almost surely convergence.

Independent defaults lead to unrealistic loss scenarios

- We conclude that the independence assumption, or more generally, the i.i.d assumption for the individual default indicators $X_1, X_2, \dots X_m$ implies that the fractional number of defaults in the portfolio $\frac{N_m}{m}$ converges to the constant p almost surely.

- It is an empirical fact, observed many times in the history, that defaults tend to cluster and $\frac{N_m}{m}$ have often values much bigger than p.

- Consequently, the empirical (i.e. observed) density for $\frac{N_m}{m}$ will have much more "fatter" tails compared with the binomial distribution.

- We will therefore next look at portfolio credit models that can produce more realistic loss scenarios, with densities for $\frac{N_m}{m}$ that have fat tails, which implies that $\frac{N_m}{m}$ does **not** converges to a constant with probability 1, when $m \to \infty$.

Conditional expectations

Before we continue this lecture, we need to introduce the concept of conditional expectations

- Let L^2 denote the space of all random variables X such that $\mathbb{E}\left[X^2\right] < \infty$
- Let Z be a random variable and let $L^2(Z) \subseteq L^2$ denote the space of all random variables Y such that $Y = g(Z)$ for some function g and $Y \in L^2$
- Note that $\mathbb{E}[X]$ is the value μ that minimizes the quantity $\mathbb{E}\left[(X-\mu)^2\right]$. Inspired by this, we define the conditional expectation $\mathbb{E}[X \mid Z]$ as follows:

Definition of conditional expectations

For a random variable Z, and for $X \in L^2$, the conditional expectation $\mathbb{E}[X \mid Z]$ is the random variable $Y \in L^2(Z)$ that minimizes $\mathbb{E}\left[(X-Y)^2\right]$.

- Intuitively, we can think of $\mathbb{E}[X \mid Z]$ as the orthogonal projection of X onto the space $L^2(Z)$, where the scalar product $\langle X, Y \rangle$ is defined as $\langle X, Y \rangle = \mathbb{E}[XY]$.

Properties of conditional expectations

For a random variable Z it is possible to show the following properties

1. If $X \in L^2$, then $\mathbb{E}[\mathbb{E}[X \mid Z]] = \mathbb{E}[X]$
2. If $Y \in L^2(Z)$, then $\mathbb{E}[YX \mid Z] = Y\mathbb{E}[X \mid Z]$
3. If $X \in L^2$, we define $\mathrm{Var}(X|Z)$ as

$$\mathrm{Var}(X|Z) = \mathbb{E}\left[X^2 \,\middle|\, Z\right] - \mathbb{E}[X \mid Z]^2$$

   and it holds that $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Z)] + \mathrm{Var}(\mathbb{E}[X \mid Z])$.

Furthermore, for an event A, we can define the conditional probability $\mathbb{P}[A \mid Z]$ as

$$\mathbb{P}[A \mid Z] = \mathbb{E}[1_A \mid Z]$$

where 1_A is the indicator function for the event A (note that 1_A is a random variable). An example: if $X \in \{a, b\}$, let $A = \{X = a\}$, and we get that $\mathbb{P}[X = a \mid Z] = \mathbb{E}\left[1_{\{X=a\}} \,\middle|\, Z\right]$.

The mixed binomial model

- The binomial model can be extended to **the mixed binomial model** which randomizes the default probability, allowing for stronger dependence.
- **The mixed binomial model** works as follows: Let Z be a random variable (discrete or continuous) and let $p(x) \in [0, 1]$ be a function such that the random variable $p(Z)$ is well-defined.
- Let $X_1, X_2, \ldots X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise.
- Conditional on Z, the random variables $X_1, X_2, \ldots X_m$ are independent and each X_i have default probability $p(Z)$, that is $\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z)$
- The economic intuition behind this randomizing of the default probability $p(Z)$ is that Z should represent some common background variable affecting all obligors in the portfolio.

The mixed binomial model, cont

- Let $F(x)$ and $\bar{p}$ be the distribution and mean of the random variable $p(Z)$, that is,
$$F(x) = \mathbb{P}\left[p(Z) \leq x\right] \quad \text{and} \quad \mathbb{E}\left[p(Z)\right] = \bar{p}. \tag{3}$$

- If for example Z is a continuous random variable on $\mathbb{R}$ with density $f_Z(z)$ then $\bar{p}$ is given by
$$\bar{p} = \mathbb{E}\left[p(Z)\right] = \int_{-\infty}^{\infty} p(z) f_Z(z) dz. \tag{4}$$

- Since $\mathbb{P}\left[X_i = 1 \,|\, Z\right] = p(Z)$ we get that $\mathbb{E}\left[X_i \,|\, Z\right] = p(Z)$, because $\mathbb{E}\left[X_i \,|\, Z\right] = 1 \cdot \mathbb{P}\left[X_i = 1 \,|\, Z\right] + 0 \cdot \left(1 - \mathbb{P}\left[X_i = 1 \,|\, Z\right]\right) = p(Z)$.

- Note that $\mathbb{E}\left[X_i\right] = \bar{p}$ and thus $\bar{p} = \mathbb{E}\left[p(Z)\right] = \mathbb{P}\left[X_i = 1\right]$ since
$$\mathbb{P}\left[X_i = 1\right] = \mathbb{E}\left[X_i\right] = \mathbb{E}\left[\mathbb{E}\left[X_i \,|\, Z\right]\right] = \mathbb{E}\left[p(Z)\right] = \bar{p}$$
where the last equality is due to (3).

The mixed binomial model, cont

- One can show that (see in the lecture notes)

$$\mathrm{Var}(X_i) = \bar{p}(1-\bar{p}) \quad \text{and} \quad \mathrm{Cov}(X_i, X_j) = \mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 = \mathrm{Var}(p(Z)) \quad (5)$$

- Next, letting all losses be the same and constant given by, say ℓ, then the total credit loss in the portfolio at time T, called L_m, is

$$L_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

thus, N_m is the **number** of defaults in the portfolio up to time T

- Again, since $\mathbb{P}\left[L_m = k\ell\right] = \mathbb{P}\left[N_m = k\right]$, it is enough to study N_m.

- Since the random variables $X_1, X_2, \ldots X_m$ now only are conditionally independent, given the outcome Z, we have

$$\mathbb{P}\left[N_m = k \,|\, Z\right] = \binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}$$

The mixed binomial model, cont.

- Hence,

$$\mathbb{P}\left[N_m = k\right] = \mathbb{E}\left[\mathbb{P}\left[N_m = k \mid Z\right]\right] = \mathbb{E}\left[\binom{m}{k} p(Z)^k (1-p(Z))^k\right] \qquad (6)$$

so if Z is a continuous random variable on $\mathbb{R}$ with density $f_Z(z)$ then

$$\mathbb{P}\left[N_m = k\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1-p(z))^{m-k} f_Z(z) dz. \qquad (7)$$

- Furthermore, because $X_1, X_2, \ldots X_m$ no longer are independent we have that

$$\text{Var}(N_m) = \text{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \text{Var}(X_i) + \sum_{i=1}^m \sum_{j=1, j\neq i}^m \text{Cov}(X_i, X_j) \qquad (8)$$

and by homogeneity in the model we thus get

$$\text{Var}(N_m) = m\text{Var}(X_i) + m(m-1)\text{Cov}(X_i, X_j). \qquad (9)$$

- So inserting (5) in (9) we get that

$$\text{Var}(N_m) = m\bar{p}(1-\bar{p}) + m(m-1)\left(\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2\right). \qquad (10)$$

The mixed binomial model, cont.

- Next, it is of interest to study how our portfolio will behave when $m \to \infty$, that is when the number of obligors in the portfolio goes to infinity.
- Recall that $\text{Var}(aX) = a^2\text{Var}(X)$ so this and (10) imply that

$$\text{Var}\left(\frac{N_m}{m}\right) = \frac{\text{Var}(N_m)}{m^2} = \frac{\bar{p}(1-\bar{p})}{m} + \frac{(m-1)\left(\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2\right)}{m}.$$

- We therefore conclude that

$$\text{Var}\left(\frac{N_m}{m}\right) \to \mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 = \text{Var}(p(Z)) \quad \text{as } m \to \infty \tag{11}$$

- Note that in the case when $p(Z)$ is a constant, say p, so that $p = \bar{p}$. we are back in the standard binomial loss model and

$$\mathbb{E}\left[p(Z)^2\right] - \bar{p}^2 = p^2 - p^2 = 0 \quad \text{so} \quad \text{Var}\left(\frac{N_m}{m}\right) \to 0 \quad \text{as} \quad m \to \infty$$

i.e. the fractional number of defaults in the portfolio converge to the constant $p = \bar{p}$ as portfolio size tend to infinity (law of large numbers.)

The mixed binomial model, cont.

- So in the mixed binomial model, we see from (11) that the law of large numbers do not hold, i.e. $\text{Var}\left(\frac{N_m}{m}\right)$ does not converge to 0.
- Consequently, the fractional number of defaults in the portfolio $\frac{N_m}{m}$ does not converge to a constant as $m \to \infty$.
- This is due to the fact that $X_1, X_2, \ldots X_m$, are **not** independent. The dependence among $X_1, X_2, \ldots X_m$ is created by Z.
- However, conditionally on Z, we have that the law of large numbers hold (because if we condition on Z, then $X_1, X_2, \ldots X_m$ are i.i.d with default probability $p(Z)$), that is

$$\text{given a "fixed" outcome of } Z \quad \text{then} \quad \frac{N_m}{m} \to p(Z) \quad \text{as} \quad m \to \infty \tag{12}$$

- Since a.s convergence implies convergence in distribution (12) implies that for any $x \in [0, 1]$ we have

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \mathbb{P}\left[p(Z) \leq x\right] \quad \text{when} \quad m \to \infty. \tag{13}$$

The mixed binomial model, cont.

- Note that (13) can also be verified intuitive from (12) by making the following observation. From (12) we have that

$$\mathbb{P}\left[\frac{N_m}{m} \leq x \,\middle|\, Z\right] \to \begin{cases} 0 & \text{if } p(Z) > x \\ 1 & \text{if } p(Z) \leq x \end{cases} \qquad \text{as } m \to \infty$$

that is,

$$\mathbb{P}\left[\frac{N_m}{m} \leq x \,\middle|\, Z\right] \to 1_{\{p(Z) \leq x\}} \qquad \text{as } \to \infty. \tag{14}$$

- Next, recall that

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] = \mathbb{E}\left[\mathbb{P}\left[\frac{N_m}{m} \leq x \,\middle|\, Z\right]\right] \tag{15}$$

so (14) in (15) renders

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \mathbb{E}\left[1_{\{p(Z) \leq x\}}\right] = \mathbb{P}\left[p(Z) \leq x\right] = F(x) \qquad \text{as } m \to \infty$$

where $F(x) = \mathbb{P}\left[p(Z) \leq x\right]$, i.e. $F(x)$ is the distribution function of the random variable $p(Z)$.

Large Portfolio Approximation (LPA)

Hence, from the above remarks we conclude the following important result:

Large Portfolio Approximation (LPA) for mixed binomial models

For large portfolios in a mixed binomial model, the distribution of the fractional number of defaults $\frac{N_m}{m}$ in the portfolio converges to the distribution of the random variable $p(Z)$ as $m \to \infty$, that is for any $x \in [0, 1]$ we have

$$\mathbb{P}\left[\frac{N_m}{m} \leq x\right] \to \mathbb{P}\left[p(Z) \leq x\right] \quad \text{when} \quad m \to \infty. \tag{16}$$

The distribution $\mathbb{P}\left[p(Z) \leq x\right]$ is called the Large Portfolio Approximation (LPA) to the distribution of $\frac{N_m}{m}$.

The above result implies that if $p(Z)$ has heavy tails, then the random variable $\frac{N_m}{m}$ will also have heavy tails, as $m \to \infty$, which then implies a strong default dependence in the credit portfolio.

Examples of mixing distributions (next two lectures)

- **Example 1:** A mixed binomial model with $p(Z) = Z$ where Z is a beta distribution, $Z \sim \text{Beta}(a, b)$ and by definition of a beta distribution it holds that $\mathbb{P}[0 \leq Z \leq 1] = 1$ so that $p(Z) \in [0, 1]$.

- **Example 2:** Another possibility for mixing distribution $p(Z)$ is to let $p(Z)$ be a logit-normal distribution. This means that

$$p(Z) = \frac{1}{1 + \exp\left(-(\mu + \sigma Z)\right)}$$

where $\sigma > 0$ and Z is a standard normal. Note that $p(Z) \in [0, 1]$.

- **Example 3:** The mixed binomial model inspired by the Merton model (will be discussed coming lectures) with $p(Z)$ given by

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{17}$$

where Z is a standard normal and $N(x)$ is the distribution function of a standard normal distribution. Furthermore, $\rho \in [0, 1]$ and $\bar{p} = \mathbb{P}[X_i = 1]$. Note that $p(Z) \in [0, 1]$.

Thank you for your attention!