Financial Risk: Credit Risk, Lecture 3

Alexander Herbertsson

Centre For Finance/Department of Economics
School of Economics, Business and Law, University of Gothenburg

E-mail: alexander.herbertsson@cff.gu.se
or
alexander.herbertsson@economics.gu.se

Financial Risk, Chalmers University of Technology,
Göteborg
Sweden

May 2, 2017

Content of Lecture

- Discussion of a mixed binomial model inspired by the Merton model

- Derive the large-portfolio approximation formula in this framework

- Discussion how to incorporate random losses in the mixed binomial loss model

Stochastic processes and the Brownian motion

- A continuous-time stochastic process $(Z_t)_{t \in [0, \infty)}$, is a collection of random variables indexed by time $t \in [0, \infty)$,

- For a given random outcome, a continuous-time stochastic process Z_t can be seen as a function of time $t \geq 0$

- Example of a continuous-time stochastic process is the Brownian motion $(W_t)_{t \geq 0}$ sometimes also denoted a Wiener process.

- The following holds for a Brownian motion $(W_t)_{t \geq 0}$

  1. $W_0 = 0$

  2. $(W_t)_{t \geq 0}$ has a continuous path with probability one

  3. For $0 \leq s < t$ then $W_t - W_s \sim N(0, t - s)$, i.e. $W_t - W_s$ is normally distributed with zero mean and variance $t - s$.

  4. $(W_t)_{t \geq 0}$ has independent increments, i.e. for any time points $0 < s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ is independent of $W_{t_2} - W_{s_2}$

Brownian motion, cont.

Different trajectories for a Brownian motion W_t
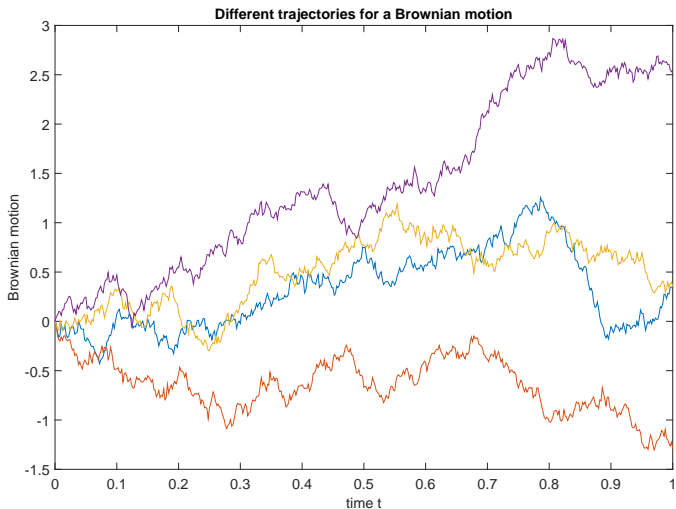
The mixed binomial model inspired by the Merton Model

- Consider a credit portfolio model, not necessary homogeneous, with m obligors, and where each obligor can default up to fixed time point, say T.

- Assume that each obligor i (think of a firm named i) follows the Merton model, i.e. the value of obligor i-s asset $V_{t,i}$ at time t follows the dynamics

$$dV_{t,i} = \mu_i V_{t,i} dt + \sigma_i V_{t,i} dB_{t,i} \tag{1}$$

where $B_{t,i}$ is a stochastic process defined as

$$B_{t,i} = \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i} \tag{2}$$

where $\rho \in [0,1]$ and $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ are independent standard Brownian motions.

- It is then possible to show that $B_{t,i}$ is also a standard Brownian motion. Hence, due to (1) we then know that $V_{t,i}$ is a GBM so by using Ito´s lemma, we get

$$V_{t,i} = V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B_{t,i}} \tag{3}$$
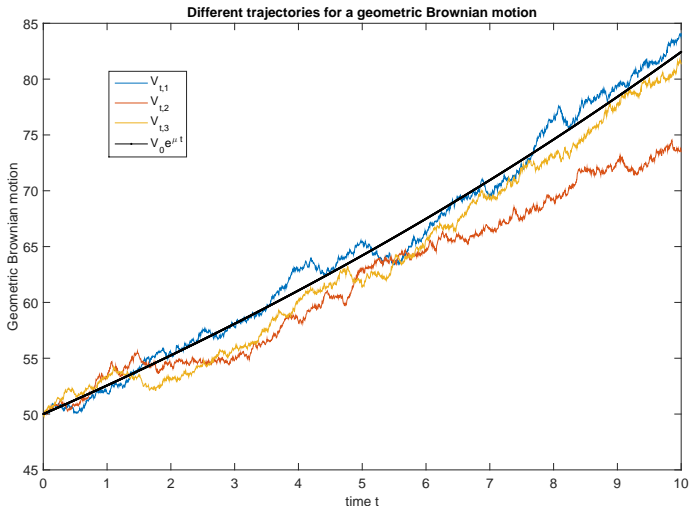
Geometric Brownian motion

Different trajectories for a Geometric Brownian motion $V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ for $V_0 = 50, \mu = 0.05, \sigma = 0.25$ (Brownian motion same as on slide 4)

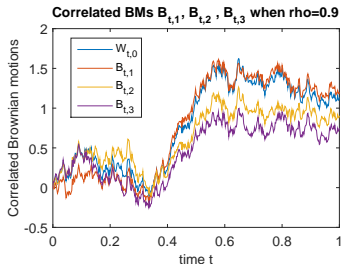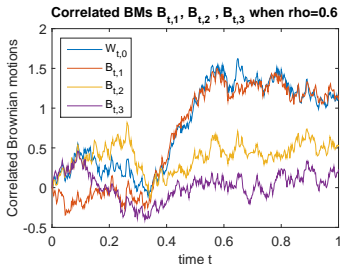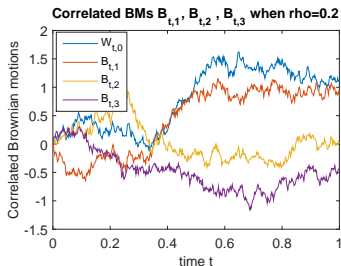Different trajectories for a geometric Brownian motion
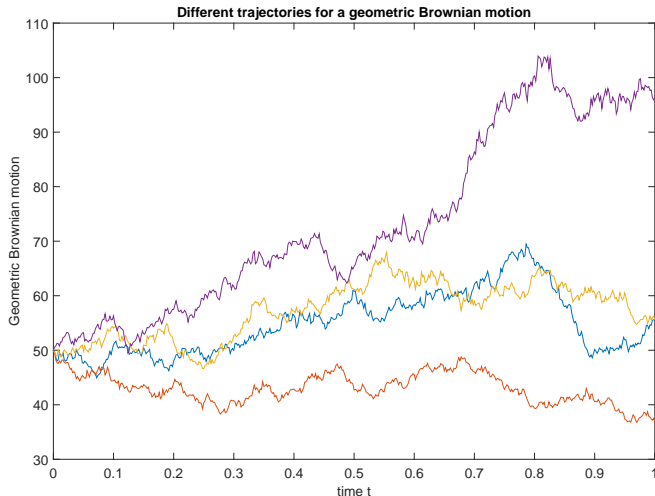
Geometric Brownian motion, cont

Different trajectories for a Geometric Brownian motion $V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ for $V_0 = 50, \mu = 0.05, \sigma = 0.025$ and the function $V_0 e^{\mu t}$

Correlated Brownian motions $B_{t,i}$, $i = 1, 2, 3$, given by (2) for different ρ
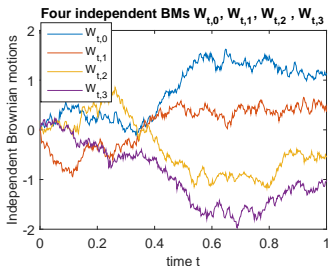
Correlated Geometrical Brownian motions $V_{t,i}$

Correlated geom. Brownian motions $V_{t,i}$ as in (3) when $B_{t,i}$ as in (2) for different ρ, and same as in prev. slide. $V_{t,i} = 50$, $\mu_i = 0.05$, $\sigma_i = 0.25$ for each $i = 1, 2, 3$

The mixed binomial model inspired by the Merton Model

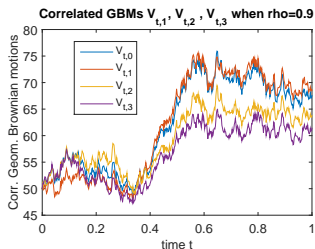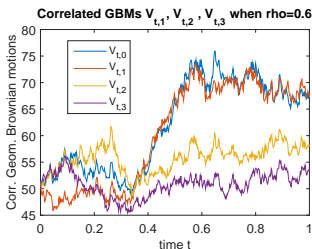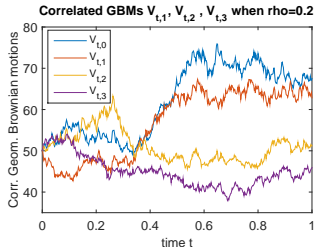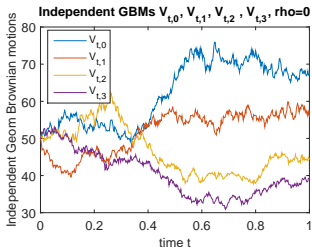- The intuition behind (1) and (2) is that the asset for each obligor i is driven by a common process $W_{t,0}$ representing the economic environment, and an individual process $W_{t,i}$ unique for obligor i, where $i = 1, 2, \ldots, m$.

- This means that the asset for each obligor i, depend both on a macroeconomic random process (common for all obligors) and an idiosyncratic random process (i.e. unique for each obligor). This will create a **dependence** among these obligors. To see this, recall that $\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$ so due to (2)

$$
\begin{aligned}
\text{Cov}(B_{t,i}, B_{t,j}) &= \mathbb{E}[B_{t,i} B_{t,j}] - \mathbb{E}[B_{t,i}]\mathbb{E}[B_{t,j}] \\
&= \mathbb{E}\left[\left(\sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i}\right)\left(\sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,j}\right)\right] \\
&= \mathbb{E}\left[\rho W_{t,0}^2\right] + \sqrt{\rho}\sqrt{1-\rho}\left(\mathbb{E}[W_{t,0} W_{t,i}] + \mathbb{E}[W_{t,0} W_{t,j}]\right) \\
&\quad + (1-\rho)\mathbb{E}[W_{t,j} W_{t,i}] \\
&= \rho\mathbb{E}\left[W_{t,0}^2\right] = \rho t
\end{aligned}
$$

where the third equality is due to $\mathbb{E}[W_{t,j} W_{t,i}] = 0$ when $i \neq j$.

The mixed binomial model inspired by the Merton Model

- Hence, $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ which implies that there is a dependence of the processes that drives the asset values $V_{t,i}$. To be more specific,

$$\text{Corr}(B_{t,i}, B_{t,j}) = \frac{\text{Cov}(B_{t,i}, B_{t,j})}{\sqrt{\text{Var}(B_{t,i})}\sqrt{\text{Var}(B_{t,i})}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho \qquad (4)$$

so $\text{Corr}(B_{t,i}, B_{t,j}) = \rho$ which is the mutual dependence among the obligors created by the macroeconomic latent variable $W_{t,0}$

- Note that if $\rho = 0$, we have $\text{Corr}(B_{t,i}, B_{t,j}) = 0$ which makes the asset values $V_{t,1}, V_{t,2}, \ldots, V_{t,m}$ independent (so the obligors are independent).

- Next, let D_i be the debt level for each obligor i and recall from the Merton model that obligor i defaults if $V_{T,i} \leq D_i$, that is if

$$V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i B_{T,i}} < D_i \qquad (5)$$

which, by using the definition of $B_{t,i}$ is equivalent with the event

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}W_{T,0} + \sqrt{1-\rho}W_{T,i}\right) < 0 \qquad (6)$$

The mixed binomial model inspired by the Merton Model

- Next, recall that for each i, $W_{i,T} \sim N(0, T)$, i.e $W_{i,T}$ is normally distributed with zero mean and variance T. Hence, if $Y_i \sim N(0,1)$, $W_{i,T}$ has the same distribution as $\sqrt{T}Y_i$ for $i = 0, 1, \ldots, m$ where $Y_0, Y_1, \ldots, Y_M$ also are independent. Furthermore, define Z as Y_0, i.e $Z = Y_0$. This in (6) yields

$$\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}\sqrt{T}Z + \sqrt{1-\rho}\sqrt{T}Y_i \right) < 0 \quad (7)$$

and dividing with $\sigma_i\sqrt{T}$ renders

$$\frac{\ln V_{0,i} - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} + \sqrt{\rho}Z + \sqrt{1-\rho}Y_i < 0. \quad (8)$$

We can rewrite the inequality (8) as

$$Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \quad (9)$$

where C_i is a constant given by

$$C_i = \frac{\ln\left(V_{0,i}/D_i\right) + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} \quad (10)$$

The mixed binomial model inspired by the Merton Model

- Hence, from the previous slides we conclude that

$$V_{T,i} < D_i \quad \text{has same prob/cond.prob as} \quad Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \qquad (11)$$

where C_i is a constant given by (10). Next define X_i as

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D_i \\ 0 & \text{if } V_{T,i} > D_i \end{cases} \qquad (12)$$

- Then (11) implies that

$$\begin{aligned}
\mathbb{P}\left[X_i = 1 \mid Z\right] = \mathbb{P}\left[V_{T,i} < D_i \mid Z\right] &= \mathbb{P}\left[Y_i < \frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \,\bigg|\, Z\right] \\
&= N\left(\frac{-\left(C_i + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right)
\end{aligned} \qquad (13)$$

where $N(x)$ is the distribution function of a standard normal distribution.

- The last equality in (13) follows from the fact that $Y_i \sim N(0,1)$ and that Y_i is independent of Z in (11).

The mixed binomial model inspired by the Merton Model

- Next, assume that all obligors in the model are identical, so that $V_{0,i} = V_0$, $D_i = D$, $\mu_i = \mu$ and thus $C_i = C$ for $i = 1, 2, \ldots, m$.

- Then we have a homogeneous static credit portfolio, where we consider the time period up to T.

- Furthermore, Equation (13) implies that

$$\mathbb{P}[X_i = 1 \mid Z] = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{14}$$

where C is a constant given by (10) with $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$ and thus $C_i = C$ for all obligors i.

- Let Z be the "economic background variable" in our homogeneous portfolio and define $p(Z)$ as

$$p(Z) = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \tag{15}$$

where $N(x)$ is the distribution function of a standard normal distribution.

The mixed binomial model inspired by the Merton Model

- Since, $p(Z) \in [0, 1]$, we would like to use $p(Z)$ in a mixed binomial model.

- To be more specific, let $X_1, X_2, \ldots X_m$ be identically distributed random variables such that $X_i = 1$ if obligor i defaults before time T and $X_i = 0$ otherwise.

- Furthermore, conditional on Z, the random variables $X_1, X_2, \ldots X_m$ are independent and each X_i have default probability $p(Z)$, that is

$$\mathbb{P}[X_i = 1 \mid Z] = p(Z) = N\left(\frac{-(C + \sqrt{\rho}Z)}{\sqrt{1 - \rho}}\right). \tag{16}$$

- We call this the mixed binomial model inspired by the Merton model or sometimes simply a mixed binomial Merton model.

The mixed binomial model inspired by the Merton Model

- Recall that the total credit loss in the portfolio at time T, called L_m, is

$$L_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where } N_m = \sum_{i=1}^{m} X_i$$

- In the mixed binomial Merton model Z is a continuous random variable on $\mathbb{R}$ so from last lecture we know that the loss distribution $F_{L_m}(x)$ is given by

$$F_{L_m}(x) = \sum_{k=0}^{\lfloor \frac{x}{\ell} \rfloor} \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1-p(z))^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz. \qquad (17)$$

where $p(u) = N\left(\frac{-(C+\sqrt{\rho}u)}{\sqrt{1-\rho}}\right)$

- However, if m is "large" we have the following approximation for the loss distribution $F_{L_m}(x) = \mathbb{P}[L_m \leq x]$

$$F_{L_m}(x) \approx F\left(\frac{x}{\ell m}\right) \quad \text{if } m \text{ is "large"}. \qquad (18)$$

for any $x \in [0, \ell m]$ and where $F(x) = \mathbb{P}[p(Z) \leq x]$.

The mixed binomial Merton model and LPA, cont.

- We therefore next want to find an explicit expression of $F(x)$ where $F(x) = \mathbb{P}[p(Z) \leq x]$. From (16) we know that $p(Z) = N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right)$ where Z is a standard normal random variable, i.e. $Z \sim N(0, 1)$.

- Hence, $F(x) = \mathbb{P}[p(Z) \leq x] = \mathbb{P}\left[N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq x\right]$ so

$$\mathbb{P}\left[N\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq x\right] = \mathbb{P}\left[\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \leq N^{-1}(x)\right]$$

$$= \mathbb{P}\left[-Z \leq \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right]$$

$$= N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)$$

where the last equality is due to
$\mathbb{P}[-Z \leq x] = \mathbb{P}[Z \geq -x] = 1 - \mathbb{P}[Z \leq -x]$ and $1 - N(-x) = N(x)$ for any x, due to the symmetry of a standard normal random variable.

The mixed binomial Merton model and LPA, cont.

- Hence, $F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)$ so what is left is to find C.

- Since our model is inspired by the Merton model, we have that

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D \\ 0 & \text{if } V_{T,i} > D \end{cases} \tag{19}$$

so $\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D]$. However, from (8) and (11) we conclude that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D] = \mathbb{P}\left[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C\right] \tag{20}$$

where C is given by Equation (10) in the homogeneous case where $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$, $\mu_i = \mu$ and consequently $C_i = C$ for $i = 1, 2, \ldots, m$.

Furthermore, since Z and Y_i are standard normals then $\sqrt{\rho}Z + \sqrt{1-\rho}Y_i$ will also be standard normal. Hence, $\mathbb{P}\left[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq -C\right] = N(-C)$ and this observation together with (20) implies that

$$\mathbb{P}[X_i = 1] = \mathbb{P}[V_{T,i} < D] = N(-C). \tag{21}$$

The mixed binomial Merton model and LPA, cont.

- Recall that $\bar{p} = \mathbb{E}[p(Z)] = \int_0^1 p(z) f_Z(z) dz$ so $\bar{p} = \mathbb{P}[X_i = 1]$ since $\mathbb{P}[X_i = 1 \mid Z] = p(Z)$ and thus

$$\mathbb{P}[X_i = 1] = \mathbb{E}[\mathbb{P}[X_i = 1 \mid Z]] = \mathbb{E}[p(Z)] = \bar{p}$$

- Hence, from (21) we have $\bar{p} = N(-C)$ so

$$C = -N^{-1}(\bar{p}) \tag{22}$$

which means that we can ignore C (and thus also ignore V_0, D, σ and μ, see (10)) and instead directly work with the default probability $\bar{p} = \mathbb{P}[X_i = 1]$. Hence, we estimate $\bar{p}$ to 5%, say, which then implicitly defines the quantizes V_0, D, σ and μ via (10) and (22).

- Finally, going back to $F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right)$ and using (22) we conclude that

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p})\right)\right) \tag{23}$$

where $F(x) = \mathbb{P}[p(Z) \leq x]$.

The mixed binomial Merton model and LPA, cont.

- Hence, if m is large enough, we can in the mixed binomial model inspired by the Merton model, use (32) to get the following approximation for the loss distribution $F_{L_m}(x) = \mathbb{P}\left[L_m \leq x\right]$

$$\mathbb{P}\left[L_m \leq x\right] \approx N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}\left(\frac{x}{\ell m}\right) - N^{-1}\left(\bar{p}\right)\right)\right). \qquad (24)$$

where $\bar{p} = \mathbb{P}\left[X_i = 1\right]$ is the individual default probability for each obligor.

- The approximation (23) or equivalently, (24) is sometimes denoted the **LPA in a static Merton framework**, and was first introduced by Vasicek 1991, at KMV, in the paper *"Limiting loan loss probability distribution"*.

- The **LPA in a Merton framework** and its offsprings (i.e. variants) is today **widely** used in the industry (Moody's-KMV, CreditMetrics etc. etc.) for risk management of large credit/loan portfolios, especially for computing regulatory capital in **Basel II** and **Basel III** (Basel III is currently being implemented (since end of 2013)).

The mixed binomial Merton model: The role of ρ

- Recall from (4) that ρ was the correlation parameter describing the dependence between the Brownian motions $B_{t,i}$ that drives each obligor i's asset price, i.e. $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ for all $t > 0$ so $\text{Corr}(B_{t,i}, B_{t,j}) = \rho$.

- Recall that $X_1, X_2, \ldots, X_m$ was defined as in (19), that is

$$X_i = \begin{cases} 1 & \text{if } V_{T,i} < D \\ 0 & \text{if } V_{T,i} > D \end{cases}$$

  where $V_{T,i}$ is the asset given by (3) in the homogeneous case.

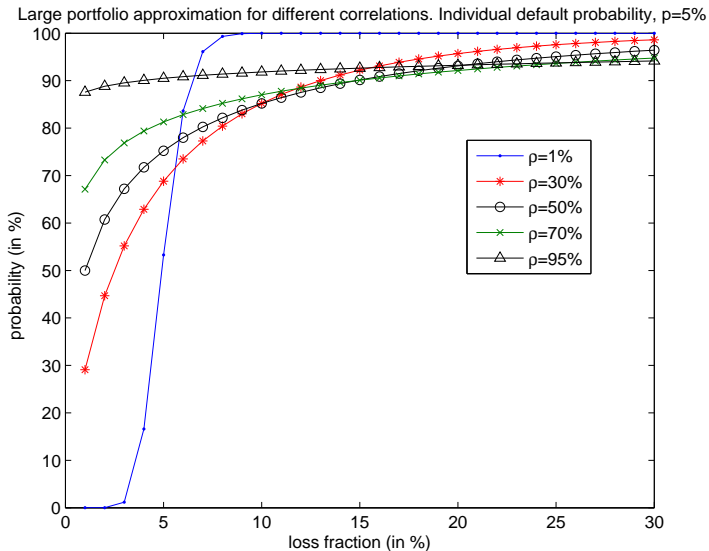- One can show that for $i \neq j$ then (see in the lecture notes for details)

$$\text{Cov}(X_i, X_j) = 0 \quad \text{if } \rho = 0 \tag{25}$$

  and

$$\text{Cov}(X_i, X_j) > 0 \quad \text{if } \rho > 0. \tag{26}$$

- We therefore conclude that ρ is a measure of **default dependence** among the zero-one variables $X_1, X_2, \ldots, X_m$ in the mixed binomial Merton model.

The mixed Merton binomial model and LPA

Large portfolio approximation for different correlations. Individual default probability, p=5%

- Given the limiting distribution $F(x)$

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) - N^{-1}(\bar{p})\right)\right) \tag{27}$$

  we can also find the density $f_{\text{LPA}}(x)$ of $F(x)$, that is $f_{\text{LPA}}(x) = \frac{dF(x)}{dx}$.
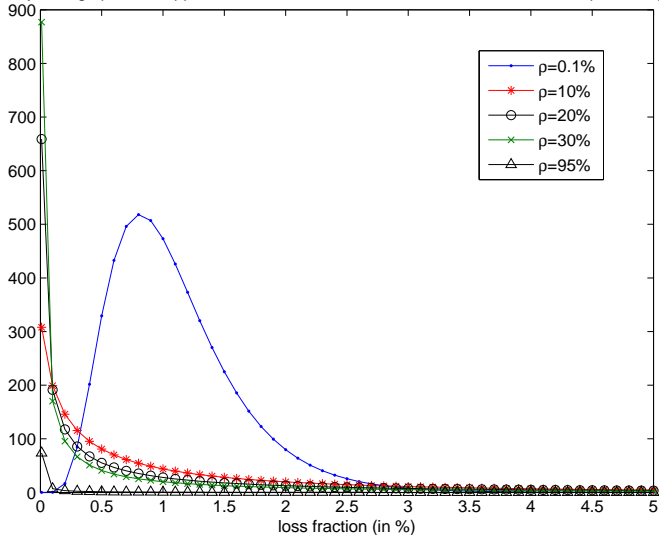
- It is possible to show that

$$f_{\text{LPA}}(x) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(\frac{1}{2}(N^{-1}(x))^2 - \frac{1}{2\rho}\left(N^{-1}(\bar{p}) - \sqrt{1-\rho}N^{-1}(x)\right)^2\right) \tag{28}$$

- This density is just an approximation, and fails for small number of the loss fraction.

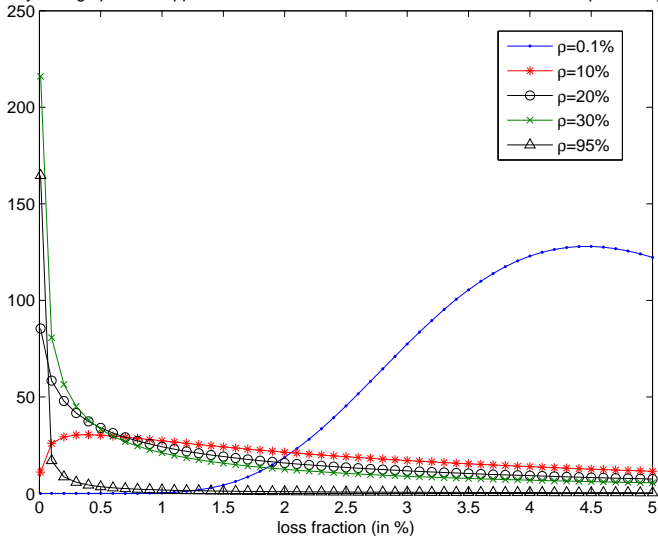Density of large portfolio approximation for different correlations. Individual default probability, p=1%

The mixed Merton binomial model and LPA, cont.

Density of large portfolio approximation for different correlations. Individual default probability, p=5%

Density of large portfolio approximation for different correlations. Individual default probability, p=5%
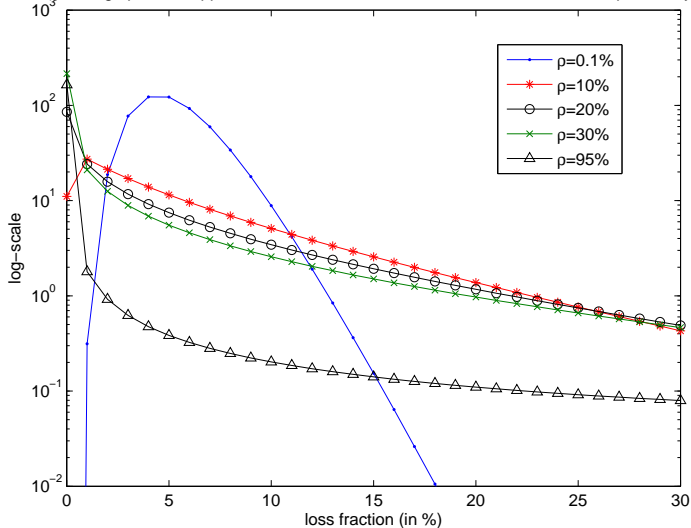
VaR in the mixed binomial Merton model

Consider a static credit portfolio with m obligors in a mixed binomial model inspired by the Merton framework where

- the individual one-year default probability is $\bar{p}$

- the individual loss is ℓ

- the default correlation is ρ

By assuming the LPA setting we can now state the following result for the one-year credit Value-at-Risk $\text{VaR}_\alpha(L)$ with confidence level $1 - \alpha$.

VaR in the mixed binomial Merton model using the LPA setting

With notation and assumptions as above, the one-year $\text{VaR}_\alpha(L)$ is given by

$$\text{VaR}_\alpha(L) = \ell \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right). \tag{29}$$

Useful exercise: Derive the formula (29).

Note that variants of the formula (29) is extensively used for computing regulatory capital in **Basel II** and **Basel III**

Random losses in the mixed binomial loss model

- In the last three lectures the individual loss ℓ_i for each obligor i have been a constant ℓ same for all obligors, when studying the mixed binomial loss model, that is $\ell = \ell_1 = \ell_2 = \ldots = \ell_m$

- It is possible to extend the mixed binomial loss models to allow for random losses ℓ_i for each obligor $i = 1, 2, \ldots, m$

- By homogeneity, the distribution of these losses must be same for all obligors, and by linearity of VaR, the losses are in percent, i.e. values in $[0, 1]$

- Let Z be the mixing distribution in a mixed binomial model with individual default probability $p(Z) = \mathbb{P}[X_i = 1 \mid Z]$ same for all obligors.

- One way to introduce random losses, is to let the individual losses $\ell_i(Z)$ be random variables which conditional on Z, are i.i.d, all having the distribution $\ell(Z)$ for some function $\ell(x) \in [0, 1]$ for all x

- Hence, conditionally on Z, then $\ell_1(Z), \ell_2(Z), \ldots, \ell_m(Z)$ are i.i.d with distribution given by $\ell(Z)$

Random losses in the mixed binomial loss model, cont.

- The portfolio loss L_m will now be given by $L_m = \sum_{i=1}^{m} \ell_i(Z)X_i$

- Depending on the nature of the individual loss distribution $\ell(Z)$ one can sometimes get closed form expressions for the exact loss distribution $F_{L_m}(x) = \mathbb{P}[L_m \leq x]$, for example if $\ell(Z)$ is a discrete distribution

- Conditionally on Z, the random variables $\ell_1(Z)X_1, \ell_2(Z)X_2, \ldots \ell_m(Z)X_m$ are i.i.d with distribution $\ell(Z)p(Z)$.

- Thus, conditionally on Z we can use the law of large numbers for $\frac{L_m}{m}$ to conclude that

    given a "fixed" outcome of Z then $\dfrac{L_m}{m} \to \ell(Z)p(Z)$ as $m \to \infty$ (30)

- Since a.s convergence implies convergence in distribution then (30) implies that for any $x \in [0,1]$ we have

$$\mathbb{P}\left[\frac{L_m}{m} \leq x\right] \to \mathbb{P}[\ell(Z)p(Z) \leq x] \quad \text{when} \quad m \to \infty. \qquad (31)$$

- We also have for any $x \in [0, \infty)$, or in fact any $x \in [0, m]$ (why?) that

$$F_{L_m}(x) = \mathbb{P}\left[L_m \leq x\right] = \mathbb{P}\left[\frac{L_m}{m} \leq \frac{x}{m}\right]$$

and this in (31) then implies that

$$F_{L_m}(x) \to \mathbb{P}\left[\ell(Z)p(Z) \leq \frac{x}{m}\right] \quad \text{as } m \to \infty$$

where we recall that $\ell(Z) \in [0, 1]$.

- Hence, if m is "large" we have the following approximation

$$F_{L_m}(x) \approx \mathbb{P}\left[\ell(Z)p(Z) \leq \frac{x}{m}\right] \quad \text{for any } x \in [0, m] \tag{32}$$

- Depending on the nature of $\ell(Z)$ one can sometimes get closed form expressions of $\mathbb{P}\left[\ell(Z)p(Z) \leq x\right]$, for example if $\ell(Z)$ is a discrete distribution

- However, we can always find an estimation of L_m by simulating the random variables Z and $X_1, \ldots, X_m$ and $\ell_1(Z), \ell_2(Z), \ldots \ell_m(Z)$

Monte-Carlo simulations

Monte-Carlo simulation of the portfolio credit loss

Let n be the number of simulations

For each $j = 1, 2, \ldots, n$, repeat the following five steps:

1. Simulate the random variable Z and compute $p(Z) \in [0, 1]$.

2. Simulate the i.i.d sequence $U_1, U_2, \ldots, U_m$ where U_i is uniformly distributed on $[0, 1]$ and independent of Z.

3. For each $i = 1, 2, \ldots, m$ define X_i as

$$X_i = \begin{cases} 1 & \text{if } U_i \leq p(Z) \\ 0 & \text{otherwise, i.e. if } U_i > p(Z) \end{cases} \tag{33}$$

4. If losses are random, simulate $\ell_1(Z), \ell_2(Z), \ldots \ell_m(Z)$

5. Compute $L_m^{(j)} = \sum_{i=1}^{m} X_i \ell_i(Z)$.

From the simulated sequence $\{L_m^{(j)}\}_{j=1}^{n}$ we can find the empirical distribution function and use it to find an estimate of Value-at-Risk etc.

Monte-Carlo simulations, cont.

Let us motivate why (33) for generating $X_1, \ldots, X_m$ implies that $\mathbb{P}[X_i = 1 \mid Z] = p(Z)$ for each $i = 1, 2, \ldots, m$.

Let $F_{U_i}(x) = x$ be the distribution function for U_i which is uniformly distributed on $[0, 1]$.

Given $p(Z)$ we then have by construction that

$$\mathbb{P}[X_i = 1 \mid Z] = \mathbb{P}[U_i \leq p(Z) \mid Z] = F_{U_i}(p(Z)) = p(Z) \qquad (34)$$

where second equality is due to fact that U_i is independent of Z. The final equality in (34) follows from $F_{U_i}(x) = x$ since U_i is uniformly distributed on $[0, 1]$.

This proves (33).

Thank you for your attention!