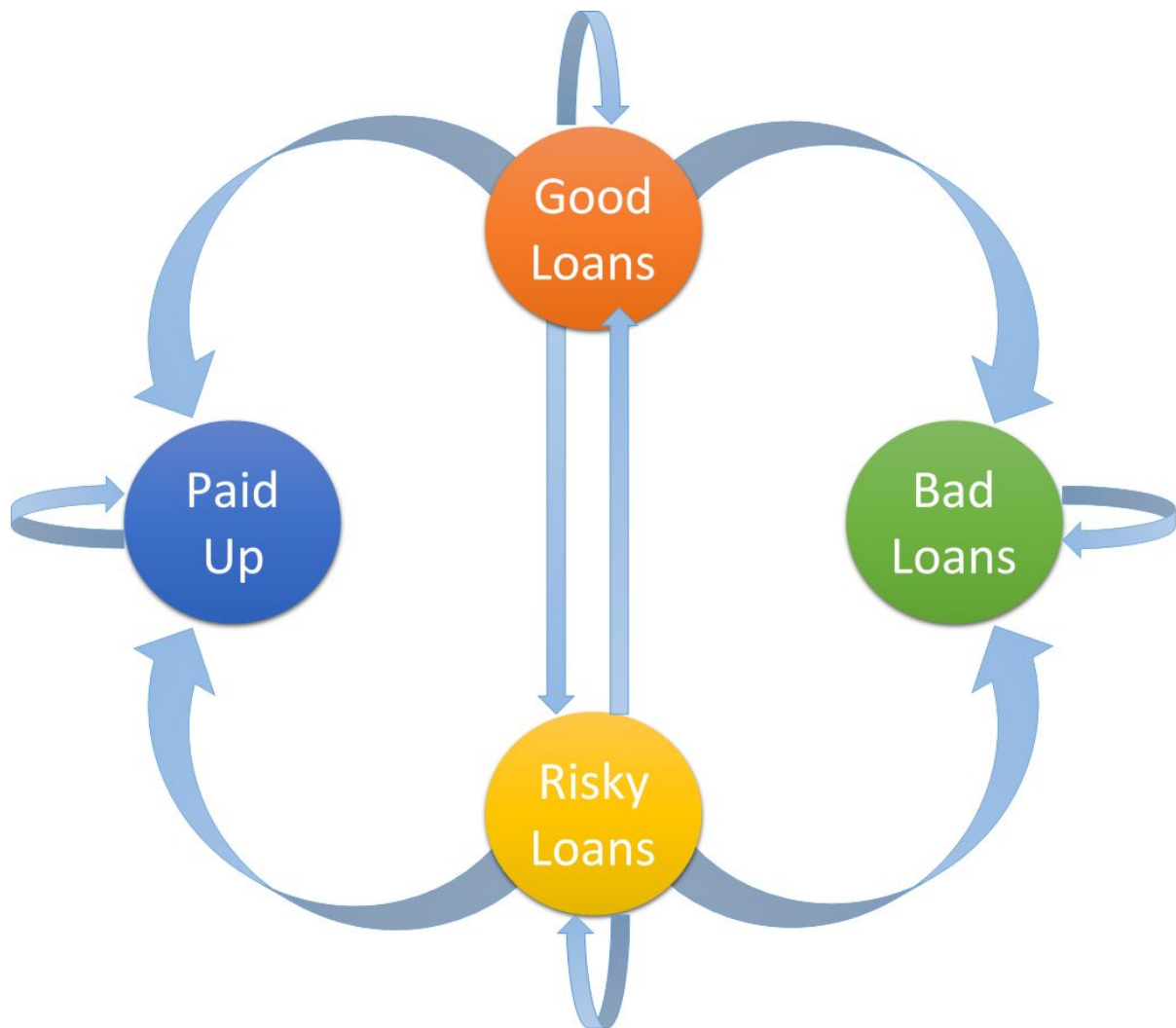


# An introduction to Markov chains and their applications within finance



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# 1. Introduction

Markov chains are an important mathematical tool in stochastic processes. The underlying idea is the Markov Property, in other words, that some predictions about stochastic processes can be simplified by viewing the future as independent of the past, given the present state of the process. This is used to simplify predictions about the future state of a stochastic process.

This report will begin with a brief introduction, followed by the analysis, and end with tips for further reading. The analysis will introduce the concepts of Markov chains, explain different types of Markov Chains and present examples of its applications in finance.

## 1.1 Background

Andrei Markov was a Russian mathematician who lived between 1856 and 1922. He was a poorly performing student and the only subject he didn't have difficulties in was mathematics. He later studied mathematics at the university of Petersburg and was lectured by Pafnuty Chebyshev, known for his work in probability theory. Markov's first scientific areas were in number theory, convergent series and approximation theory. His most famous studies were with Markov chains, hence the name and his first paper on the subject was published in 1906. He was also very interested in poetry and the first application he found of Markov chains was in fact in a linguistic analysis of Pusjkins work *Eugene Onegin*. [1]



A. A. Markov (1886).

## 1.2 Delimitations

This report will not delve too deep into the mathematical aspects of Markov chains. Instead, it will focus on delivering a more general understanding and serve as an introduction to the subject.

## 1.3 Purpose

The purpose of this report is to give a short introduction to Markov chains and to present examples of different applications within finance.

## 2. Analysis

### 2.1 Introduction to Markov chains

Markov chains are a fundamental part of stochastic processes. They are used widely in many different disciplines. A Markov chain is a stochastic process that satisfies the Markov property, which means that the past and future are independent when the present is known. This means that if one knows the current state of the process, then no additional information of its past states is required to make the best possible prediction of its future. This simplicity allows for great reduction of the number of parameters when studying such a process. [2]

In mathematical terms, the definition can be expressed as follows:

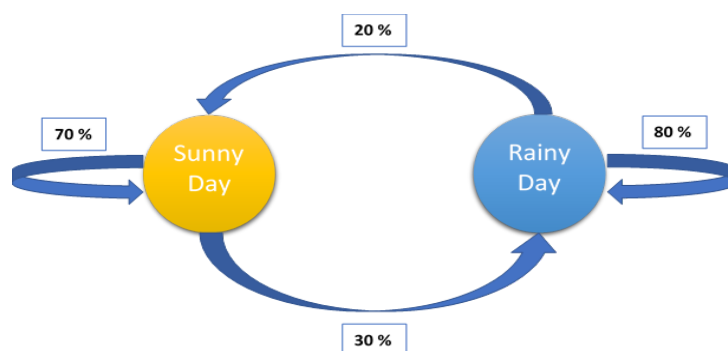
A stochastic process  $X = \{X_n, n \in N\}$  in a countable space  $S$  is a discrete-time Markov chain if:

*For all  $n \geq 0, X_n \in S$*

*For all  $n \geq 1$  and for all  $i_0, \dots, i_{n-1}, i_n \in S$ , we have :*

$$P\{X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = P\{X_n = i_n | X_{n-1} = i_{n-1}\} \quad [2]$$

Markov chains are used to compute the probabilities of events occurring by viewing them as states transitioning into other states, or transitioning into the same state as before. We can take weather as an example: If we arbitrarily pick probabilities, a prediction regarding the weather can be the following: If it is a sunny day, there is a 30% probability that the next day will be a rainy day, and a 20% probability that if it is a rainy day, the day after will be a sunny day. If it is a sunny day, there is therefore a 70% chance that the next day will be another sunny day, and if today is a rainy day, there is a 80% chance that the next day will be a rainy day as well. This can be summarized in a transition diagram, where all of the possible transitions of states are described:



To approach this mathematically one views today as the current state,  $S_0$ , which is a  $1 \times m$  vector. The elements of this vector will be the current state of the process. In our weather example, we define  $S = [Sunny \ Rainy]$ . Where  $S$  is called our state space, in which all the elements are all the possible states that the process can attain. If, for example, today is a sunny day, then the  $S_0$  vector will be  $S_0 = [1 \ 0]$ , because there is 100% chance of a sunny day and zero chance of it being a rainy day. To get to the next state, the transition probability

matrix is required, which is just the state transition probabilities summarized in a matrix. In this case it will be as follows:

$$P = \begin{matrix} & \begin{matrix} S & R \end{matrix} \\ \begin{matrix} S \\ R \end{matrix} & \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \end{matrix} \text{ or more generally in this case: } P = \begin{matrix} & \begin{matrix} S & R \end{matrix} \\ \begin{matrix} S \\ R \end{matrix} & \begin{bmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{bmatrix} \end{matrix}$$

To get to the next state,  $S_1$ , you simply calculate the matrix product  $S_1 = S_0P$ . Since calculations for successive states of S is only of the type  $S_n = S_{n-1}P$ , the general formula for computing the probability of a process ending up in a certain state is  $S_n = S_0P^n$ . This allows for great simplicity when calculating the probabilities far into the future. For example, if today is a sunny day then the state vector 120 days from now,  $S_{120}$ , is  $S_{120} = [0.4 \ 0.6]$ . [3]

## 2.2 Explanation of different concepts regarding Markov chains

When approaching Markov chains there are two different types; discrete-time Markov chains and continuous-time Markov chains. This means that we have one case where the changes happen at specific states and one where the changes are continuous. In our report we will mostly focus on discrete-time Markov chains.

One example to explain the discrete-time Markov chain is the price of an asset where the value is registered only at the end of the day. The value of the Markov chain in discrete-time is called the state and in this case the state corresponds to the closing price. A continuous-time Markov chain changes at any time. This can be explained with any example where the measured events happens at a continuous time and lacks “steps” in its appearance. One well known example of continuous-time Markov chain is the poisson process, which is often practised in queuing theory. [1]

For a finite Markov chain the state space S is usually given by  $S = \{1, \dots, M\}$  and the countably infinite state Markov chain state space usually is taken to be  $S = \{0, 1, 2, \dots\}$ . These different variances differ in some ways that will not be referred to in this paper. [4]

A Markov chain can be stationary and therefore be independent of the initial state in the process. This phenomenon is also called a steady-state Markov chain and we will see this outcome in the example of market trends later on, where the probabilities for different outcomes converge to a certain value. However, an infinite-state Markov chain does not have to be steady state, but a steady-state Markov chain must be time-homogenous. Which by definition means that the transition probabilities matrix  $P_{i,j}(n, n + 1)$  is independent of n. [3]

## 2.3 Application areas of Markov chains

Since Markov chains can be designed to model many real-world processes, they are used in a wide variety of situations. These fields range from the mapping of animal life populations to search-engine algorithms, music composition and speech recognition. In economics and finance, they are often used to predict macroeconomic situations like market crashes and cycles between recession and expansion. Other areas of application include predicting asset and option prices, and calculating credit risks. When considering a continuous-time financial

market Markov chains are used to model the randomness. The price of an asset, for example, is set by a random factor – a stochastic discount factor – which is defined using a Markov chain [5].

## 2.4 Application of Markov chains to credit risk measurement

In the application of Markov chains to credit risk measurement, the transition matrix represents the likelihood of the future evolution of the ratings. The transition matrix will describe the probabilities that a certain company, country, etc. will either remain in their current state, or transition into a new state. [6] An example of this below:

Table 1. The transition probability table from Standard & Poor's (1999).

Asset/Year	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.9193	0.0746	0.0048	0.0008	0.0004	0.0000	0.0000	0.0000
AA	0.6400	0.9181	0.0676	0.0060	0.0006	0.0012	0.0003	0.0000
A	0.0700	0.0227	0.9169	0.0512	0.0056	0.0025	0.0001	0.0004
BBB	0.0400	0.0270	0.0556	0.8788	0.0483	0.0102	0.0017	0.0024
BB	0.0400	0.0010	0.0061	0.0775	0.8148	0.0790	0.0111	0.0101
B	0.0000	0.0010	0.0028	0.0046	0.0695	0.8280	0.0396	0.0545
CCC	0.1900	0.0000	0.0037	0.0075	0.0243	0.1213	0.6045	0.2369
D	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

[6]

The main problem in this application is determining the transition matrix. Of course, these probabilities could be estimated by analysing historical data from credit rating agencies, such as Standard & Poor, Moody's and Fitch. This can, though, lead to unreliable numbers in case the future does not develop as smoothly as in the past. It can therefore be more reliable to base the estimations on a combination of empirical data and more subjective, qualitative data such as opinions from experts. This is because the market view is a mixture of beliefs determined by both historical ratings and a more extreme view of the ratings. To combine different sources of information in this way, one may use credibility theory. Actuarial credibility theory provides a consistent and convenient way of how to combine information, and how to weigh the different data sources. [6]

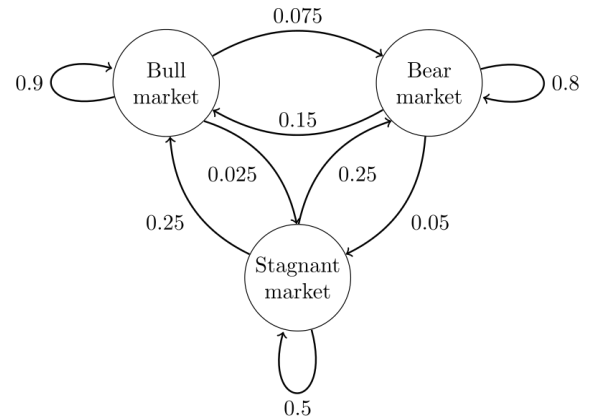
Another problem with deciding the transition matrix is that maybe it is not appropriate to use a homogenous Markov chain to model credit risk over time. This is because it does not capture the time-varying behaviour of the default risk. Of course, a non-homogeneous model could be more realistic - but on the other hand much more complicated to use. [6]

## 2.5 Markov chains to predict market trends

Markov chains and their respective diagrams can be used to model the probabilities of certain financial market climates and thus predicting the likelihood of future market conditions [7]. These conditions, also known as trends, are:

- Bull markets: *periods of time where prices generally are rising, due to the actors having optimistic hopes of the future.*
- Bear markets: *periods of time where prices generally are declining, due to the actors having a pessimistic view of the future.*
- Stagnant markets: *periods of time where the market is characterized by neither a decline nor rise in general prices.*

In fair markets, it is assumed that the market information is distributed equally among its actors and that prices fluctuate randomly. This means that every actor has equal access to information such that no actor has an upper hand due to inside-information. Through technical analysis of historical data, certain patterns can be found as well as their estimated probabilities. [7] For example, consider a hypothetical market with Markov properties where historical data has given us the following patterns: After a week characterized of a bull market trend there is a 90% chance that another bullish week will follow. Additionally, there is a 7.5% chance that the bull week instead will be followed by a bearish one, or a 2.5% chance that it will be a stagnant one. After a bearish week there's an 80% chance that the upcoming week also will be bearish, and so on. By compiling these probabilities into a table, we get the following transition matrix  $M$ :



<i>From</i>	<i>To</i>	Bull	Bear	Stagnant
Bull		0.9	0.075	0.025
Bear		0.15	0.8	0.05
Stagnant		0.25	0.25	0.5

$$= \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = M$$

We then create a 1x3 vector  $C$  which contains information about which of the three different states any current week is in; where column 1 represents a bull week, column 2 a bear week and column 3 a stagnant week. In this example we will choose to set the current week as bearish, resulting in the vector  $C = [0 \quad 1 \quad 0]$ .

Given the state of the current week, we can then calculate the possibilities of a bull, bear or stagnant week for any number of  $n$  weeks into the future. This is done by multiplying the vector  $C$  with the matrix, giving the following:

$$\text{One week from now: } C * M^1 = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^1 = [0.15 \quad 0.8 \quad 0.05]$$

$$\text{5 weeks from now: } C * M^5 = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^5 = [0.48 \quad 0.45 \quad 0.07]$$

$$\text{52 weeks from now: } C * M^{52} = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{52} = [0.63 \quad 0.31 \quad 0.05]$$

$$\text{99 weeks from now: } C * M^{99} = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{99} = [0.63 \quad 0.31 \quad 0.05]$$

From this we can conclude that as  $n \rightarrow \infty$ , the probabilities will converge to a steady state, meaning that 63% of all weeks will be bullish, 31% bearish and 5% stagnant.

What we also see is that the steady-state probabilities of this Markov chain do not depend upon the initial state [3]. The results can be used in various ways, some examples are calculating the average time it takes for a bearish period to end, or the risk that a bullish market turns bearish or stagnant.

### 3. Summary

Markov chains are an important concept in stochastic processes. They can be used to greatly simplify processes that satisfy the Markov property, namely that the future state of a stochastic variable is only dependent on its present state. This means that knowing the previous history of the process will not improve the future predictions - which of course significantly reduces the amount of data that needs to be taken into account.

Mathematically, Markov chains consist of a state space, which is a vector whose elements are all the possible states of a stochastic variable, the present state of the variable, and the transition matrix. The transition matrix contains all the probabilities that the variable will transition from one state to another, or remain the same. To calculate the probabilities of a variable ending up in certain states after  $n$  discrete partitions of time, one simply multiplies the present state vector with the transition matrix raised to the power of  $n$ .

There are different types of concepts regarding Markov chains depending of the nature of the parameters and application areas. They can be computed over discrete or continuous time. The state space can vary to be finite or countably infinite and depending on which, behave in different ways. A Markov chain with a countably infinite state space can be stationary which means that the process can converge to a steady state.

Markov chains are used in a broad variety of academic fields, ranging from biology to economics. When predicting the value of an asset, Markov chains can be used to model the randomness. The price is set by a random factor which can be determined by a Markov chain.

Regarding the application to credit risk measurement, calculating the transition matrix is the most important part when applying Markov chains in this subject. One way that has been presented in this paper is to use a combination of empirical data from several years in the past from credible credit rating institutions (Standard & Poor, Moody's, Fitch), and other types of more qualitative data. One should bear in mind that the homogenous Markov chain model probably will be less realistic than a non-homogenous model, but much less complicated.

By analyzing the historical data of a market, it is possible to distinguish certain patterns in its past movements. From these patterns, Markov diagrams can then be formed and used to predict future market trends as well as the risks associated with them.

## 4. Further Reading

[1] A. Andersson, *Introduktion till Markovkedjor: Med diskret och kontinuerlig tid*. Göteborg: HB matematiklitteratur i Göteborg, 2004.

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