Markowitz portfolio theory

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1 Introduction

A portfolio is the set of assets that an investor chooses to invest in. Choosing the optimal portfolio is a complex problem, requiring one to make predictions about current state of the assets under consideration and to define what the optimal is. In Harry Markowitz's 1952 paper *Portfolio Selection*, he seeks to solve the second part of the problem: finding the optimal portfolio once predictions about the state of the assets have been made.

2 Markowitz portfolio theory

The simplest way to define the optimal portfolio is the one that maximizes the return. If one were to construct a portfolio based on this idea alone, however, the portfolio would consist of the single asset with the highest return. There are some obvious issues with this method, since this single asset may perform worse than expected and thereby destroy the success of the entire portfolio. Markowitz's solution is to define a set of efficient portfolios from which an optimal portfolio for each individual can be chosen. An efficient portfolio is one that maximizes the expected return while minimizing the variance, which is the chosen measure of risk. Specifically, an efficient portfolio has the lowest variance for its expected return and the highest return for its variance [2].

Given N assets, the expected return and the variance for the portfolio as a whole is

$$E = \sum_{i=1}^{N} X_i \mu_i, \; V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X_j,$$

where X_i is the portion of the portfolio invested in asset i, μ_i is the expected return of asset i, and σ_{ij} is the covariance between the returns of asset i and j [2]. The X_i are chosen by the investor, and $\sum_{i=1}^{N} X_i = 1$. We will also assume that $X_i \geq 0$. μ_i and σ_{ij} have to be calculated based on knowledge of the assets before any decisions about the portfolio can be made.

3 Examples with two and three assets

In the case with two assets, x_1 and x_2, we consider the case where the first one has a higher expected return and risk. The assets may have some of correlation, but for this case, it is relatively low, such that a portfolio with a combination of the two can result in a lower variance. As seen in figure 1, the minimum variance occurs at $x_1 \approx 0.25$ and is thus on the set of efficient portfolios. The set of efficient portfolios is thereby all portfolios with $x_1 \gtrsim 0.25$, since all of these values have an expected return that cannot be beaten while holding the variance constant and a variance that cannot be lowered without lowering the expected return.

Now consider a case with three assets:

$$X_1 \geq 0, \; X_2 \geq 0, \; X_3 = 1 - X_1 - X_2 \geq 0$$

A representation of the portfolio is shown in figure 2. The result is similar to the two asset case, but the variance is now shown with isovariance curves, and the expected return is shown to increase in the direction orthogonal to the isomean lines. For some combination of the three assets, the variance is at the minimum. This is at the point x at the center of the variance curves. x is also the beginning of the efficient portfolios. The rest of the efficient portfolios have to have higher returns than x, so the next part of the efficient portfolios is the line in the direction of increasing returns orthogonal to the variance curves starting from the point x. Once the edge of the attainable set is reached, the efficient portfolios follow this edge in the direction where the returns increase until a corner is reached.

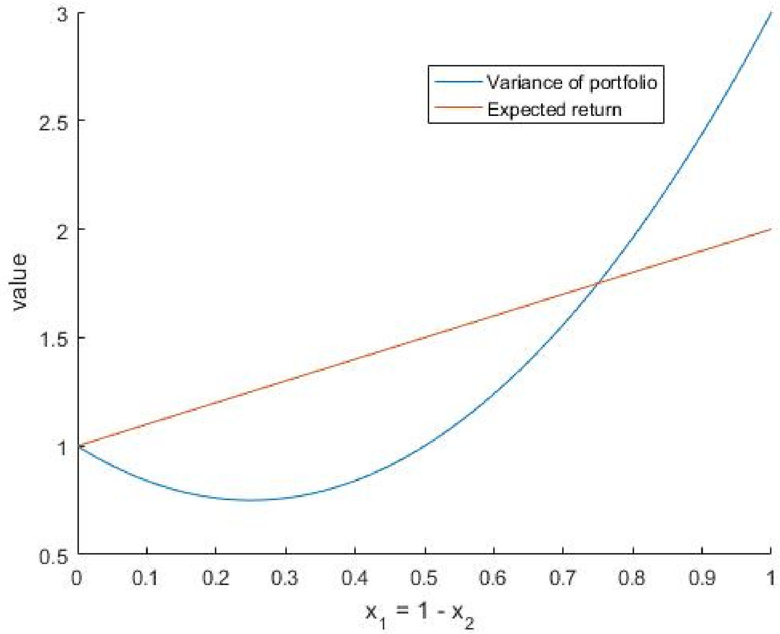

Figure 1: A representation of how the expected return and variance can depend on the ratio of assets in a two asset portfolio. The values on the y-axis are arbitrary.

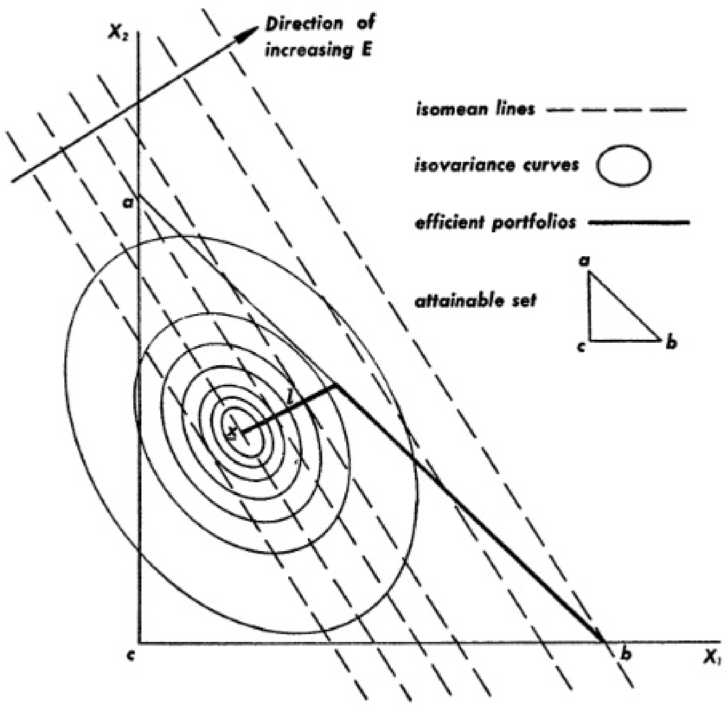

Figure 2: A representation of a portfolio with three assets. This figure is taken from Markowitz's paper [2].

4 Results from the theory

Markowitz's portfolio theory shows that diversification, owning a variety of assets, leads to lower risk and is, in general, preferable. However, the assets have to be sufficiently different from one another, corresponding to a low covariance, for the diversification to have much effect in reducing risk. The more assets included in the portfolio, the more the total variance can be reduced if assets with low correlation are chosen. With many assets, the individual variances become unimportant compared to the correlation between the assets. On the other hand, if the portfolio consists of only a few assets, the individual variances are more important than the correlations [3].

A simple result from the theory is clear when considering two different portfolios in the form of two different mutual funds. If the two mutual funds have the same variance, then a portfolio that consists of a combination of the two will have a lower variance so long as the two funds are not perfectly correlated. This scenario is shown in figure 3.

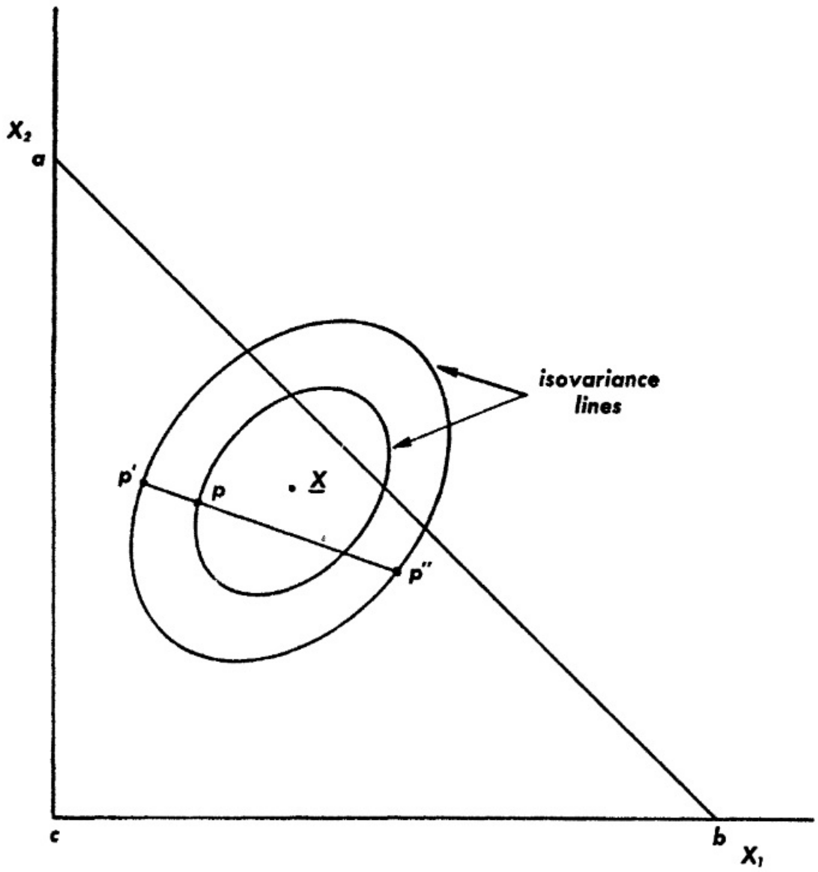

Figure 3: Illustration of the difference between two portfolios in a three asset scenario. p' and p" are two portfolios with equal variance, and p is a linear combination of the two with lower variance. The return of p is a weighted average of the returns of p' and p". This figure is taken from Markowitz's paper [2].

5 Application of the theory

In practice, there are hundreds of assets available to investors looking to create a portfolio. Trying to show how the variance and expected return of the portfolio changes when the ratio of the different assets change graphically becomes impossible at this scale. The efficient portfolios can, however, be found using optimization methods. The problem can further be simplified by choosing how risky the portfolio should be, either by specifying a variance or by deciding how much greater the return has to be for increased risk to be worth it and minimizing the variance minus the expected return times this riskiness ratio.

The main issue with applying Markowitz's portfolio theory is that one has to make predictions about the expected return of each asset and the covariance between each pair of assets being considered in the portfolio. This means that there are, for a portfolio with N assets, $2N + \binom{N}{2}$ parameters that have to be calculated before the theory can be applied. One can make predictions about the values of these parameters based on historical data, but this introduces the problem of how long of a period of historical data is needed before it can be considered representative of the future [1]. It is likely that quite a long period of years is needed to estimate performance. Even with extensive historical data, the question of its applicability to the future remains.

Another problem with following Markowitz's portfolio theory is that systematic risk is not handled. Systematic risk is a type of risk that affects most, if not all, assets. For example, "inflation, interest rates, unemployment levels, exchange rates, or Gross National Product-levels" [1] are all contributors of systematic risk. It is, however, hard to think of a portfolio that would not have these types of risks, so this is not a problem specific to Markowitz's portfolio theory.

In addition, Markowitz's portfolio theory does not consider transaction costs when determining what portfolios are efficient. If one type of asset has higher transaction costs than another, then it may not be optimal to include those assets in the portfolio, despite them being part of an efficient portfolio in theory. It may, however, be relatively simple to modify the expected return to account for the transaction costs as well.

6 Conclusions

The Markowitz's portfolio theory cannot be directly applied to real world decisions, since it requires such a large amount of knowledge about all of the available assets. The knowledge about efficient portfolios and that in general a combination of assets is, in general, better then the individual ones alone are valuable results obtained from this theory.

The main strategy to reduce risks in the future is diversification. Owning a variety of different kinds of assets makes the total portfolio less likely to sustain large losses. In addition, accurate assessments of the expected returns and variances of the assets under consideration are needed in order to correctly account for risks.

7 Reading guide

For those that want to know more about Markowitz's portfolio theory and portfolio management in general, we recommend the following:

- A short video describing a efficient portfolio, *Efficient Portfolio Frontier*, by Option Alpha:

  `https://www.youtube.com/watch?v=3ntwyjXZdS0&feature=related`

- The paper where Harry Markowitz introduces his theory, *Portfolio Selection*:

  `https://www.math.ust.hk/~maykwok/courses/ma362/07F/markowitz_JF.pdf`

- A paper more focused on the modern portfolio theory, *A simplified perspective of the Markowitz portfolio theory*, by Myles E. Mangram:

  `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2147880`

- The book Portfolio Selection: Efficient diversification of investments by Harry Markowitz, which fully explains Markowitz's theory.
- A rather long but simple lecture about portfolio management, *Portfolio Management*, from MIT OpenCourseWare: `https://www.youtube.com/watch?v=8TJQhQ2GZ0Y`

References

[1] Myles E. Mangram. "A Simplified Perspective of the Markowitz Portfolio Theory". In: *Global Journal of Business Research* 7.1 (2013). URL: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2147880.

[2] Harry Markowitz. "Portfolio Selection". In: *The Journal of Finance* 7.1 (1952), pp. 77–91. URL: https://www.math.ust.hk/~maykwok/courses/ma362/07F/markowitz_JF.pdf.

[3] Harry Markowitz. *Portfolio Selection: Efficient Diversification of Investments*. Cowles Foundation, 1959.