MVE220 ECERCISE SESSION #2

Block maxima

Generalized extreme value distribution (GEV)

$$G(x) = \exp\left\{-\left(1+\gamma \, \frac{x-\overset{\text{location}}{\mu}}{\underset{\text{scale}}{\sigma}}\right)^{-1/\underset{\text{shape}}{\gamma}}\right\}$$

Special case: $\gamma = 0$ (Gumbel dist.)

$$G(x) = \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}$$

If $M_n = \max\{\underbrace{X_1, \dots, X_n}_{\text{i.i.d. RVs}}\}$, then

$$P(M_n \leq x) = P(X_1 \leq x, \dots, X_n \leq x) = F(x)^n$$

Peaks over threshold

Generalized Pareto distribution (GP)

$$H(x) = 1 - \left(1 + \frac{\gamma}{\sigma} x\right)_+^{-1/\gamma} \quad \left(= 1 - e^{-x/\sigma} \text{ if } \gamma = 0\right)$$

Distribution of exceedances over threshold u:

$$F_u(x) = P(X - u \leq x \mid X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}$$

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4) $P\left(\frac{M_n - a_n}{b_n} \leq x\right) = P(M_n \leq b_n x + a_n) = P(X_1 \leq b_n x + a_n, \dots, X_n \leq b_n x + a_n)$

$$\underset{\substack{X_1, X_2, \dots \\ \text{independent}}}{=} \prod_{i=1}^{n} P(X_i \leq b_n x + a_n) \underset{\substack{X_1, X_2 \dots \\ \text{i.d.}}}{=} \left(1 - e^{-(b_n x + a_n)/\sigma}\right)^n$$

We know that $\left(1 + \frac{a}{n}\right)^n \xrightarrow[n \to \infty]{} e^a$

so $\left(1 - \frac{e^{-x}}{n}\right)^n \rightarrow e^{-e^{-x}}$

We therefore want to find a_n, b_n s.t.

$$e^{-\frac{(b_n x + a_n)}{\sigma}} = \frac{e^{-x}}{n}$$

$$\Leftrightarrow \quad -\frac{(b_n x + a_n)}{\sigma} = -x - \ln(n)$$

$$\Leftrightarrow \quad b_n x + a_n = \sigma x + \sigma \ln(n)$$

$$\Leftrightarrow \quad b_n = \sigma \quad \text{and} \quad a_n = \sigma \ln(n).$$

7) $(X-u) \sim GP(\sigma,\gamma)$, i.e. $P(X>u+y \mid X>u) = \left(1+\frac{\gamma y}{\sigma}\right)^{-1/\gamma}$

$$\frac{1-F(u+y)}{1-F(u)}$$

Therefore, $1-F(u+y) = \left(1+\frac{\gamma y}{\sigma}\right)^{-1/\gamma}(1-F(u))$

Now

$$P(X > y+u+v \mid X > u+v) = \frac{1-F(u+y+v)}{1-F(u+v)}$$

$$= \frac{\left(1+\frac{\gamma(y+v)}{\sigma}\right)^{-1/\gamma}(1-F(u))}{\left(1+\frac{\gamma v}{\sigma}\right)^{-1/\gamma}(1-F(u))}$$

$$= \frac{\left(1+\frac{\gamma(y+v)}{\sigma}\right)^{-1/\gamma}}{\left(1+\frac{\gamma v}{\sigma}\right)^{-1/\gamma}} = \left(\frac{1+\frac{\gamma y}{\sigma}+\frac{\gamma v}{\sigma}}{1+\frac{\gamma v}{\sigma}}\right)^{-1/\gamma}$$

$$= \left(1+\frac{\gamma y}{\sigma+\gamma v}\right)^{-1/\gamma}$$

So excesses of threshold $u+v$ have the distribution

$GP(\tilde{\sigma},\tilde{\gamma}) = GP(\sigma+\gamma v, \gamma)$.

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9) $M_5 = \max\{X_1, \dots, X_5\}$. $M_5 \sim GEV(\mu, \sigma, \gamma)$.

$$P(M_5 \le x) = \exp\left(-\left(1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\gamma}\right)$$

$$M_{20} = \max\{M_5^{(1)}, \dots, M_5^{(4)}\}$$

$$P(M_{20} \le x) = P(M_5 \le x)^4 = \left(\exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\gamma}\right\}\right)^4$$

$$= \exp\left\{-4\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\gamma}\right\}$$

$$= \exp\left\{-\left(1+4^{-\gamma}-1+4^{-\gamma}\gamma\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\gamma}\right\}$$

$$= \exp\left\{-\left[1+\gamma\left(\frac{4^{-\gamma}-1}{\gamma} + \frac{(x-\mu)}{4^{\gamma}\sigma}\right)\right]^{-1/\gamma}\right\}$$

$$= \exp\left\{-\left[1+\gamma\left(\frac{x-\left(\mu+\frac{\sigma}{\gamma}(4^{\gamma}-1)\right)}{4^{\gamma}\sigma}\right)\right]^{-1/\gamma}\right\}$$

so

$$M_{20} \sim GEV(\tilde{\mu}, \tilde{\sigma}, \tilde{\gamma}) = GEV\left(\mu+\frac{\sigma}{\gamma}(4^{\gamma}-1),\ 4^{\gamma}\sigma,\ \gamma\right).$$

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13) X_1, X_2 independent RVs, $GEV(\mu = 2.1, \sigma = 1.3, \gamma = 0)$

$$G(x) = e^{-e^{-\left(\frac{x-2.1}{1.3}\right)}}$$

$M = \max(X_1, X_2)$.

$$P(M \le x) \underset{\substack{\uparrow \\ \text{i.i.d.}}}{=} P(X_1 \le x)^2 = e^{-2e^{-\frac{(x-2.1)}{1.3}}} = e^{-e^{-\left(\frac{x-2.1}{1.3} - \ln(2)\right)}}$$

$$= e^{-e^{-\left(\frac{x-(2.1+1.3\ln(2))}{1.3}\right)}}$$

so

$$M \sim GEV(\tilde{\mu}, \tilde{\sigma}, \tilde{\gamma}) = GEV(\overbrace{2.1+1.3\ln(2)}^{\approx 3.00}, 1.3, 0).$$

14) $(\hat{\mu}, \hat{\sigma}, \hat{\gamma}) = (-1.64, 0.27, -0.084)$

95% confidence interval for $\hat{\sigma}$:

$$\hat{\sigma} \pm z_{\frac{\alpha}{2}} \sqrt{0.000652} = 0.27 \pm 1.96\sqrt{0.000652}$$

$$= 0.27 \pm 0.05$$

$$= (0.22, 0.32)$$