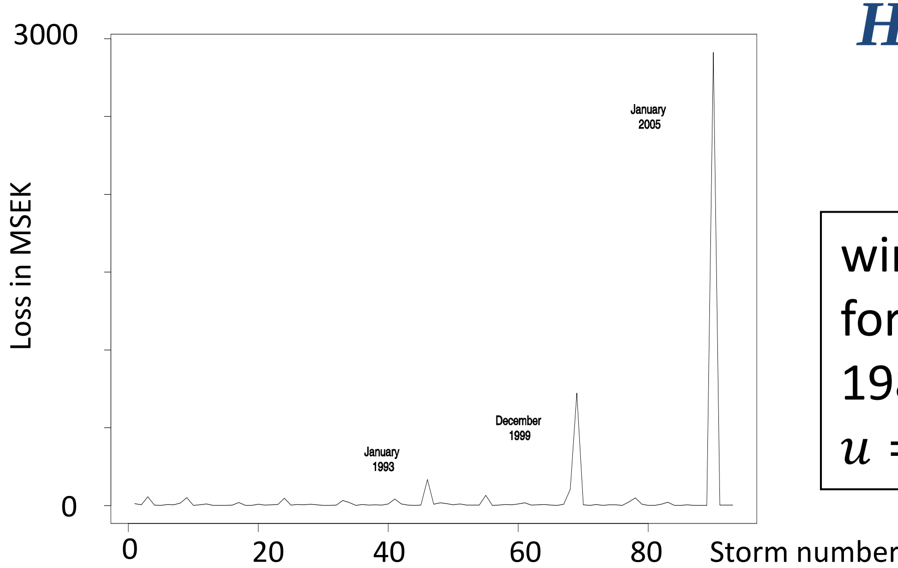
Financial Risk
4-rd quarter 2017/18

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

The Peaks over Thresholds (PoT) method *(Coles p. 74-91, H&L p. 256-259)*



windstorm losses for Länsförsäkringar 1982 – 2005: excesses of u = 1.5 MSEK

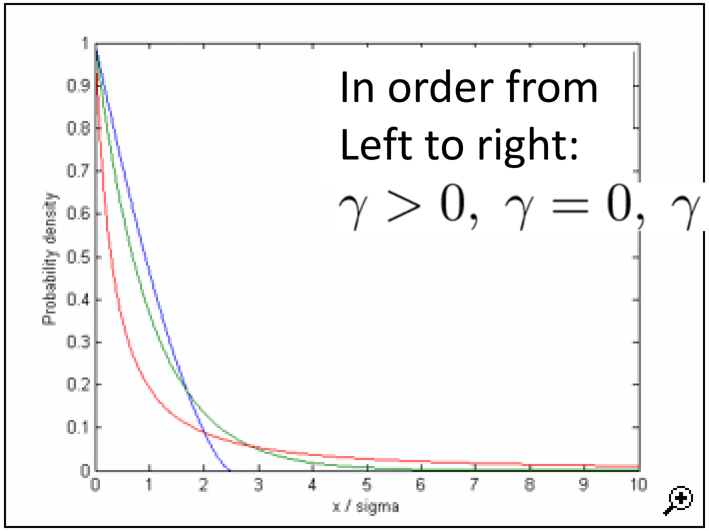
Times of exceedance of high threshold u Poisson process, excess losses ($= \text{loss} - u$) follow a Generalized Pareto (GP) distribution with distribution function

$$H(x) = 1 - \left(1 + \tfrac{\gamma}{\sigma} x\right)_+^{-1/\gamma} \quad (= 1 - e^{-x/\sigma} \text{ if } \gamma = 0),$$

exceedance times and excess sizes are all mutually independent

The choice of threshold an "art", aided by graphics: parameter stability; median excess; goodness of fit; plots

The Generalized Pareto distribution



density function of Generalized Pareto distribution

$$h(x) = \frac{d}{dx} H(x) = \frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma} x\right)_{+}^{-1/\gamma - 1} \quad \left(= \frac{1}{\sigma} e^{-x/\sigma} \ \text{ if } \ \gamma = 0\right)$$

$\gamma \geq 0$ the distribution has left endpoint *0* and right endpoint ∞

$\gamma < 0$ the distribution has left endpoint *0* and right endpoint $\sigma / |\gamma|$

the distribution is “heavytailed” for $\gamma > 0$. Then moments of order greater than $1/\gamma$ are infinite/don’t exist, exactly as for the Generalized Extreme Value distribution

The Generalized Pareto distribution

Assume the random variable *X* has d.f. *F* and let *u* be a (high) level
The distribution of exceedances then is the conditional distribution of *X-u* given that *X* is larger than *u, i.e.* it has d.f.

$$F_u(x) = P\big(X - u \le x | X > u\big) = \frac{P\big(X-u\le x \text{ and } X>u\big)}{P\big(X>u\big)} = \frac{F(x+u)-F(u)}{1-F(u)}$$

(and hence $\bar{F}_u(x) = 1 - F_u(x) = \frac{\bar{F}(x+u)}{\bar{F}(u)}$).

Mathematics similar to the one which motivated the Block Maxima Method shows that if $F_u(x)$ has a limit as the level $u \to \infty$ then this limit must be a GP distribution, *and* that the GP distribution is the only family of distributions which is stable under a change of levels (as specified in the next exercise).

Exercise: Show that if *F(x)* is a GP distribution, then also $F_u(x)$ is a GP distribution, and express the parameters of $F_u(x)$ in terms of the parameters of *F(x)*. (Treat $\gamma \neq 0$ and $\gamma = 0$ separately.)

The Poisson process

Model for times of occurrence of events which occur "randomly" in time, with a constant "intensity", e.g radioactive decay, times when calls arrive to a telephone exchange, times when traffic accidents occur ... (all during periods of stationarity)

Can be mathematically described as a counting process *N(t) = #events in [0, t]*

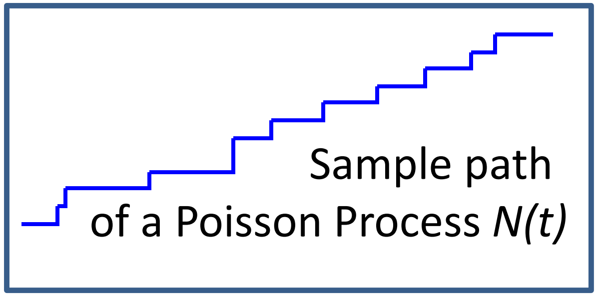
Mathematically, the counting process *N(t)* is a Poisson process if

a) The numbers of events which occur in disjoint time intervals are mutually independent

b) *N(s+t) – N(s)* has a Poisson distribution for any *s, t* $\geq$ *0, i.e.*

$$P\big(N(s+t) - N(s) = k\big) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad \text{for any} \quad s, t \geq 0, \; k = 1, 2, \ldots$$

λ is the "intensity" parameter. One interpretation of it is that λ is the expected number of events in any time interval of length *1.*



A connection between the PoT and Block Maxima methods

Suppose the PoT model holds. Thus values larger than u occur according to a Poisson process with intensity λ , this process is independent of the sizes of the excesses, and these are i.i.d. and have a GP distribution $H(x) = 1 - \left(1 + \frac{\gamma}{\sigma}x\right)_+^{-1/\gamma}$. $M_T =$ the maximum in the time interval $[1, T]$. Then

$$
\begin{aligned}
P\big(M_T \le u + x\big) &= \sum_{k=0}^{\infty} P\big(M_T \le u + x, \text{ there are k exceedances in } [0,\text{ T}]\big) \\
&= \sum_{k=0}^{\infty} H(x)^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \sum_{k=0}^{\infty} (1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \exp\{(1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})\lambda T\} \exp\{-\lambda T\} \\
&= \exp\{-(1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma} \lambda T\} \\
&= \exp\{-(1 + \gamma \frac{x - ((\lambda T)^\gamma - 1)\sigma/\gamma}{\sigma(\lambda T)^\gamma})_+^{-1/\gamma}\}
\end{aligned}
$$

Tail and quantile estimation when the underlying variables (e.g. daily portfolio losses) and not just big values themselves (e.g. large windstorm losses) are at the center of interest

Suppose we have observed the (random) number *N(u)* of exceedances by $X_1, \ldots, X_n$ of the threshold u. Writing $\bar{F}(x) = 1 - F(x)$ for the probability that an observation is larger than x, the ratio $N(u)/n$ is a natural estimator of $\bar{F}(u)$. Assume further that we have computed estimators $\hat{\sigma}, \hat{\gamma}$ of the parameters σ, γ in the GP distribution from the excesses of u. Since

$$\bar{F}(x) = \bar{F}(u)\frac{\bar{F}(x)}{\bar{F}(u)} = \bar{F}(u)\bar{F}_u(x-u),$$

a natural estimator of the "tail function" $\bar{F}(x)$, for $x > u$, then is

$$\hat{\bar{F}}(x) = \frac{N(u)}{n}(1 + \hat{\gamma}\frac{x-u}{\hat{\sigma}})^{-1/\hat{\gamma}}.$$

Solving $\hat{\bar{F}}(x_p) = p$ for x_p we get an estimator of the $1-p$-th quantile of X:

$$\hat{x}_p = u + \frac{\hat{\sigma}}{\hat{\gamma}}((\frac{n}{N(u)}p)^{-\hat{\gamma}} - 1).$$

(Why all this trouble? Why not just estimate $\bar{F}(x)$ by $N(x)/n$? Because if x is a very high level then $N(x)$ is very small or zero, and then this estimator is useless -- and it is such very large x-es we are interested in.)