Solutions to

Exam: Finansiell Risk, MVE 220/MSA400
Thursday, May 31 2018, 8:30-12:30
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Allowed material: List of Formulas, Chalmers allowed calculator.

Problems 1-4: Multiple choice, only hand in table with answers
Only one correct answer. Correct answer gives 5 points, no answer ("don't know") gives 0 points and wrong answer gives -1 point (more than one answer automatically gives -1 point).

Uppgift	a	b	c	d	e	f (Don't know)	Points
1	☐	☐	☐	☐	x	☐	
2	x	☐	☐	☐	☐	☐	
3	☐	☐	x	☐	☐	☐	
4	☐	☐	☐	x	☐	☐	

Problems 5-10: Hand in full solutions

5.) $X_1, \dots X_n$ observations of $GP(\sigma, 0)$

Density $\frac{d}{dx}\left(1-e^{-X/\sigma}\right) = \frac{1}{\sigma} e^{-X/\sigma}$

a) log likelihood function is

$$l(\sigma) = \sum_{i=1}^{n} \log\left(\frac{1}{\sigma} e^{-X_i/\sigma}\right)$$

$$= -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} X_i$$

$$\frac{d}{d\sigma} l(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} X_i = 0$$

$\Rightarrow$

$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is the ML estimator

b) $$\frac{d^2}{d\sigma^2} l(\sigma) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} \sum_{i=1}^{n} X_i$$

estimated variance of $\hat{\sigma}$ is

$$-\frac{1}{\frac{d^2}{d\sigma^2} l(\hat{\sigma})} = \frac{1}{\frac{2}{\hat{\sigma}^3} \sum_{i=1}^{n} X_i - \frac{n}{\hat{\sigma}^2}}$$

$$= \frac{\hat{\sigma}^2}{n}$$

6.) Monthly maxima independent with c.d.f $F(X) = \exp\{-(1+\frac{\gamma}{\sigma}(x-\mu))^{-1/\gamma}\}$

a) M_{12} = yearly maxima

$$P(M_{12} \le u) = \left[\exp\{-(1+\tfrac{\gamma}{\sigma}(x-\mu))^{-1/\gamma}\}\right]^{12}$$

$$= \exp\{-12(1+\tfrac{\gamma}{\sigma}(x-\mu))^{-1/\gamma}\}$$

$$P(M_{12} > u) = 1 - \exp\{-12(1+\tfrac{\gamma}{\sigma}(x-\mu))^{-1/\gamma}\}$$

b) p-th quantile, x_p, of M_{12} satisfies

$$p = P(M_{12} \le x_p) = \exp\{-12(1+\tfrac{\gamma}{\sigma}(x_p-\mu))^{-1/\gamma}\}$$

$$\log p = -12(1+\tfrac{\gamma}{\sigma}(x_p-\mu))^{-1/\gamma}$$

$$x_p = \frac{\sigma}{\gamma}\left[\left(\frac{1}{12}\log\frac{1}{p}\right)^{-\gamma} - 1\right] + \mu$$

7.) Assume wind storms come as Poisson process with intensity λ per year

X = loss in one windstorm

$X - 0.9$ = excess loss $\sim GP(\sigma, \gamma)$

M = maximum yearly loss

$$\hat{\sigma} = 3.87, \quad \hat{\gamma} = 0.70, \quad \hat{\lambda} = \frac{46}{1993-1981} = 3.83$$

$$P(M \le x) = \exp\left\{-\lambda\left(1+\gamma\frac{x-0.9}{\sigma}\right)^{-1/\gamma}\right\}$$

formula in slides

so $P(M > 850)$ is estimated by

$$1 - \exp\left\{-3.83\left(1+0.70\,\frac{850-0.9}{3.87}\right)^{-1/0.70}\right\}$$

$$= 0.029$$

7 b) Conditional distribution of excesses of 850 MSEK in one wind-storm is

$$P(X-850 \le x \mid X>850) = 1-P(X-850>x \mid X>850)$$

$$\underset{x \ge 0}{=} 1-P((X-0.9)>x+849.1 \mid X>850)$$

$$= 1-\frac{P((X-0.9)>x+849.1)}{P(X-0.9>849.1)}$$

$$= 1-\frac{P((X-0.9)>x+849.1 \mid X>0.9)}{P(X-0.9>849.1 \mid X>0.9)}$$

$$= 1-\frac{\left(1+\frac{\gamma}{\sigma}(x+849.1)\right)^{-1/\gamma}}{\left(1+\frac{\gamma}{\sigma}849.1\right)^{-1/\gamma}}$$

To find the median of the conditional excess loss distribution one hence have to solve

$$0.5 = 1-\frac{\left(1+\frac{0.70}{3.87}(x+849.1)\right)^{-1/0.70}}{\left(1+\frac{0.70}{3.87}849.1\right)^{-1/0.70}}$$

This gives $x =$ and hence the median of the conditional loss distribution is $850+532.9=1382.9$

Task 8: Financial risk, 2018-05-31

For $i = 1, 2, .., m$ let

$$X_i = \begin{cases} 1 & \text{if obligor } 1 \text{ defaults within one year} \\ 0 & \text{otherwise} \end{cases}$$

and let Z be a random variable such that

$$p(Z) = P[X_i = 1 | Z]$$

and let

$$F(x) = P[p(Z) \leq x] \quad \text{for } x \in [0,1]$$

Define N_m as $N_m = \sum_{i=1}^{m} X_i$, (i.e. N_m is the number of defaults in the portfolio within one year) and let $L_m = l \cdot N_m$ be the total portfolio credit loss within one year where l is the individual credit loss which by linearity of VaR w.l.o.g is in percent, i.e $l \in [0,1]$.
Then, the LPA-theory gives that,

$$P\left[\frac{N_m}{m} \leq x\right] \to F(x) \quad \text{as } m \to \infty$$

for $x \in [0,1]$

8.2

Hence, we have

$$F_{L_m}(x) = P[l \cdot N_m \leq x] = P\left[\frac{N_m}{m} \leq \frac{x}{l \cdot m}\right] \to F\left(\frac{x}{l \cdot m}\right)$$

as $m \to \infty$, so if m is "large" we have

$$F_{L_m}(x) \approx F\left(\frac{x}{l \cdot m}\right) \qquad (1)$$

and if $F(z)$ is a continuous random variable then (1) implies that

$$F^{\leftarrow}_{L_m}(y) \approx l \cdot m \cdot F^{-1}(y)$$

since $y = F_{L_m}(x) \Leftrightarrow x = F^{\leftarrow}_{L_m}(y) \qquad (2)$

and $y = F\left(\frac{x}{l \cdot m}\right) \Leftrightarrow x = l \cdot m \cdot F^{-1}(y) \qquad (3)$

so (1)-(3) then gives that

$$F^{\leftarrow}_{L_m}(y) \approx l \cdot m F^{-1}(y) \qquad (4)$$

when m is large. By definition we have

$$VaR_{\alpha}(L_m) = F^{\leftarrow}_{L_m}(\alpha) \qquad (5)$$

So (4) in (5) implies (when m is "large")

$$VaR_{\alpha}(L_m) \approx l \cdot m \cdot F^{-1}(\alpha) \qquad (6)$$

8.3

In the mixed binomial model inspired by the Merton framework we have that

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right)\right)$$

So to find $F^{-1}(\cdot)$ we solve for x in the equation $y = F(x)$ which gives

$$y = F(x) \iff y = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right)\right)$$

$$\iff N^{-1}(y) = \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right)$$

$$\iff N^{-1}(x) = \frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}$$

$$\iff x = N\left(\frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right)$$

Hence

$$F^{-1}(y) = N\left(\frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right) \qquad (7)$$

So (6) and (7) then implies that

$$\mathrm{VaR}_{\alpha}(L_m) \approx l \cdot m \cdot N\left(\frac{\sqrt{\rho}\, N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right)$$

when m is large

From ~~task~~ 8, or the lecture notes, we know that

$$VaR_{\alpha}(L) = l \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right) \quad (1)$$

where we here know the values of m, l and α, ~~but~~ not $\bar{p}$ and ρ.

Let $a = 118.5 \cdot 10^6$ and $b = 260.7 \cdot 10^6$

We know that

$$VaR_{95\%}(L) = a \quad (2)$$

$$VaR_{99\%}(L) = b \quad (3)$$

We want to compute $VaR_{99.9\%}(L)$.

Note that (1) & (2) implies that

$$a = l \cdot m \cdot N\left(\frac{\sqrt{\rho}N^{-1}(0.95) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right)$$

$$\Leftrightarrow \sqrt{1-\rho}\,N^{-1}\left(\frac{a}{l \cdot m}\right) - \sqrt{\rho}\,N^{-1}(0.95) = N^{-1}(\bar{p}) \quad (4)$$

In the same way, by using (1) and (3) we get that

$$\sqrt{1-p}\,N^{-1}\left(\frac{b}{l\cdot m}\right)-\sqrt{p}\,N^{-1}(0.99)=N^{-1}(\bar{p}) \quad (5)$$

So (4) and (5) implies that

$$\sqrt{1-p}\underbrace{\left(N^{-1}\left(\frac{a}{l\cdot m}\right)-N^{-1}\left(\frac{b}{l\cdot m}\right)\right)}_{=c}=\sqrt{p}\underbrace{\left(N^{-1}(0.95)-N^{-1}(0.99)\right)}_{=d}$$

$$\Leftrightarrow \sqrt{1-p}\cdot c=\sqrt{p}\cdot d$$

where $c=N^{-1}\left(\frac{a}{l\cdot m}\right)-N^{-1}\left(\frac{b}{l\cdot m}\right)$ (6)

$d=N^{-1}(0.95)-N^{-1}(0.99)$ (7)

So $\sqrt{1-p}\cdot c=\sqrt{p}\,d \Leftrightarrow (1-p)c^2=pd^2$

$$\Leftrightarrow c^2=(c^2+d^2)p \Leftrightarrow p=\frac{c^2}{c^2+d^2} \quad (8)$$

(9.3)

To compute $\mathrm{VaR}_{99.9\%}(L)$ we also need $N^{-1}(\bar{p})$, but given (8) we can use either (4) or (5) to get a numerical value for $N^{-1}(\bar{p})$.

So with

$$a = 118.5 \cdot 10^6$$
$$b = 260.7 \cdot 10^6$$
$$l = 0.6 \cdot 10^6$$
$$m = 1000$$

we get that

$$c \approx -0.68566$$
$$d = -0.68149$$
$$p = 0.5030$$
$$N^{-1}(\bar{p}) = -1.7662 \quad \left(\text{so } \bar{p} = 0.03868\right)$$

and thus

$$\mathrm{VaR}_{99.9\%}(L) = 436.17 \cdot 10^6 \text{ SEK}$$

Answer: $\mathrm{VaR}_{99.9\%}(L) = 436.2$ million SEK

Task 10: Financial risk, 2018-05-31 (10.1)

Let $L_m = 10^6 \cdot l \cdot N_m$

$a_0 = a \cdot 10^6$

$b_0 = b \cdot 10^6$

where $a = 40$ and $b = 110$.

We want to use the LPA-formula to compute $P[a_0 < L_m < b_0]$. Note that

$$P[a_0 < L_m < b_0] = P[a \cdot 10^6 < 10^6 \cdot l \cdot N_m < b \cdot 10^6]$$

$$= P\left[\frac{a}{lm} < \frac{N_m}{m} < \frac{b}{l \cdot m}\right] = \left\{ \begin{array}{l} \frac{N_m}{m} \text{ is approximately} \\ \text{a cont. rand. variable} \\ \text{when } m \text{ is large} \end{array} \right\}$$

$$= P\left[\frac{a}{lm} < \frac{N_m}{m} \leq \frac{b}{lm}\right] =$$

$$= P\left[\frac{N_m}{m} \leq \frac{b}{lm}\right] - P\left[\frac{N_m}{m} \leq \frac{a}{lm}\right] \approx F\left(\frac{b}{lm}\right) - F\left(\frac{a}{lm}\right)$$

where $F(x) = P[p(Z) \leq x]$

Hence, we have that

$$P[a_0 < L_m < b_0] \approx F\left(\frac{b}{\ell m}\right) - F\left(\frac{a}{\ell m}\right) \quad (1)$$

where $F(x) = P[p(Z) \leq x]$.

In our case $Z \sim N(0,1)$ and $p(x)$ is given by

$$p(x) = \frac{1}{1+e^{-\mu-\sigma x}}$$

Note that $p(x)$ is strictly increasing since

$$p'(x) = \frac{d}{dx}\left(\left(1+e^{-\mu-\sigma x}\right)^{-1}\right) =$$

$$= (-1)\left(-\sigma e^{-\mu-\sigma x}\right)\left(1+e^{-\mu-\sigma x}\right)^{-2} = \frac{\sigma e^{-\mu-\sigma x}}{\left(1+e^{-\mu-\sigma x}\right)^2} > 0$$

so $p'(x) > 0$ for all $x \in \mathbb{R}$.

Hence, since $p(x)$ is strictly increasing and continuous, we know that $p^{-1}(x)$ exsist and is well-defined

(10.3)

Furthermore, since $p(x)$ is strictly increasing we also have that

$$F(x) = P[p(Z) \le x] = P[Z \le p^{-1}(x)] = N(p^{-1}(x))$$

So

$$F(x) = N(p^{-1}(x)) \quad (2).$$

Next we find an expression for $p^{-1}(x)$.
To find $p^{-1}(\cdot)$, solve for x in the equation $y = p(x) \Leftrightarrow x = p^{-1}(y)$.
So, for $y \in (0,1)$ we have that

$$y = p(x) \Leftrightarrow y = \frac{1}{1+e^{-\mu-\sigma x}} \Leftrightarrow 1+e^{-\mu-\sigma x} = \frac{1}{y}$$

$$\Leftrightarrow e^{-\mu-\sigma x} = \frac{1-y}{y} \Leftrightarrow -\mu-\sigma x = \ln\left(\frac{1-y}{y}\right)$$

$$\Leftrightarrow x = -\frac{1}{\sigma}\left(\ln\left(\frac{1-y}{y}\right)+\mu\right)$$

Hence, $p^{-1}(y) = -\frac{1}{\sigma}\left(\ln\left(\frac{1-y}{y}\right)+\mu\right)$

or, $p^{-1}(x) = \frac{1}{\sigma}\left(\ln\left(\frac{x}{1-x}\right)-\mu\right) \quad (3)$

Thus, (1) and (2) implies that (10.4)

$$P[a_0 < L_m < b_0] \approx N\left(p^{-1}\left(\frac{b}{lm}\right)\right) - N\left(p^{-1}\left(\frac{a}{lm}\right)\right) \quad (4)$$

where $p^{-1}(x)$ is given by (3).

So with $a = 40$, $b = 110$, $l = 0.6$, $m = 1000$

$\sigma = 1.2399$, $\mu = -2.6371$

we get

$$p^{-1}\left(\frac{b}{lm}\right) = p^{-1}\left(\frac{40}{600}\right) = p^{-1}\left(\frac{1}{15}\right) \overset{(3)}{\approx} -0.00157862$$

$$p^{-1}\left(\frac{a}{l \cdot m}\right) = p^{-1}\left(\frac{110}{600}\right) = p^{-1}\left(\frac{11}{60}\right) \overset{(3)}{\approx} 0.92199$$

Hence, $N\left(p^{-1}\left(\frac{b}{l \cdot m}\right)\right) = N(-0.00157862) \approx 0.49937$ (5)

$$N\left(p^{-1}\left(\frac{a}{lm}\right)\right) = N(0.92199) \approx 0.82173 \quad (6)$$

so (5) & (6) in (4) finally gives that

$$P[a_0 < L_m < b_0] \approx 0.3224 = 32.24\,\%$$

Answer: 32.24%