Exam: Finansiell Risk, MVE 220/MSA400
Friday, August 24, 2018, 14:00-18:00
Jour: Ivar Simonsson, ankn 5325

Allowed material: List of Formulas, Chalmers allowed calculator.

Problems 1-4: Multiple choice, only hand in table with answers
Only one correct answer. Correct answer gives 5 points, no answer ("don't know") gives 0 points and wrong answer gives -1 point (more than one answer automatically gives -1 point).

Uppgift	a	b	c	d	e	f (Don't know)	Points
1	☐	☐	☒	☐	☐	☐	
2	☐	☒	☐	☐	☐	☐	
3	☐	☒	☐	☐	☐	☐	
4	☐	☐	☐	☐	☒	☐	

Problems 5-10: Hand in full solutions

Q5) $\hat{\mu} = -1.64$ $\widehat{V(\hat{\mu})} = 0.0141$

97.5% quantile of standard normal distribution is 1.96

95% confidence interval is

$(-1.64 - 1.96\sqrt{0.0141}, -1.64 + 1.96\sqrt{0.0141})$

$= (-1.71, 1.57)$

Q6) a) for $x > u$

$$\bar{F}(x) = \bar{F}(u)\bar{F}_u(x-u) = \bar{F}(u)\left(1+\frac{\gamma}{\sigma}(x-u)\right)^{-1/\gamma}$$

$\bar{F}(u)$ is estimated by number of exceedances divided by number of observations $= 0.05$

Inserting this and the estimated values of γ and σ gives that $\bar{F}(0.05)$ is estimated by

$0.0179 = 1.8\%$

Q7) As in lecture slides

Task 8: Financial risk, 2018-08-24

Let $L_m = 10^6 \cdot l \cdot N_m$

$a_0 = a \cdot 10^6$

$b_0 = b \cdot 10^6$

where $a = 40$ and $b = 90$.

We want to use the LPA-formula to compute $P[a_0 < L_m < b_0]$. Note that

$$P[a_0 < L_m < b_0] = P[a \cdot 10^6 < 10^6 \cdot l \cdot N_m < b \cdot 10^6]$$

$$= P\left[\frac{a}{l \cdot m} < \frac{N_m}{m} < \frac{b}{l \cdot m}\right] = \left\{\begin{array}{l} \frac{N_m}{m} \text{ is approximately} \\ \text{a continuous random} \\ \text{variable when } m \text{ is large} \end{array}\right\}$$

$$= P\left[\frac{a}{l \cdot m} < \frac{N_m}{m} \leq \frac{b}{l \cdot m}\right] =$$

$$= P\left[\frac{N_m}{m} \leq \frac{b}{l \cdot m}\right] - P\left[\frac{N_m}{m} \leq \frac{a}{l \cdot m}\right] \approx F\left(\frac{b}{lm}\right) - F\left(\frac{a}{lm}\right)$$

where $F(x) = P[p(Z) \leq x]$ and

$p(Z) = P[X_i = 1 | Z]$ with $N_m = \sum_{i=1}^{m} X_i$

Hence, we have that

$$P[a_0 < L_m < b_0] \approx F\left(\frac{b}{l\cdot m}\right) - F\left(\frac{a}{l\cdot m}\right) \quad (1)$$

where $F(x) = P[p(Z) \leq x]$

In our case $Z \sim N(0,1)$ and $p(Z)$ is given by $p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}\, Z}{\sqrt{1-\rho}}\right)$

and from Eq (3.2.2) in the formula sheet we have that

$$P[p(Z) \leq x] = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}(x) - N^{-1}(\bar{p})\right)\right)$$

so for any c we have that

$$F\left(\frac{c}{l\cdot m}\right) = P\left[p(Z) \leq \frac{c}{l\cdot m}\right] = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}\left(\frac{c}{l\cdot m}\right) - N^{-1}(\bar{p})\right)\right)$$

that is

$$P[L_m \leq c] = P\left[\frac{N_m}{m} \leq \frac{c}{l\cdot m}\right] \approx F\left(\frac{c}{l\cdot m}\right) \quad (2)$$

where

$$F\left(\frac{c}{l\cdot m}\right) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}\left(\frac{c}{l\cdot m}\right) - N^{-1}(\bar{p})\right)\right) \quad (3)$$

We know that $p = 19\%$ but we don't know $\bar{p}$. However we know that 8.3

$$P\left[L_m \leq d \cdot 10^6\right] = \alpha \quad (4)$$

where

$$d = 23 \text{ and } \alpha = 0.565 \quad (5)$$

So (2), (3), (4) then implies that

$$\alpha = N\left(\frac{1}{\sqrt{p}}\left(\sqrt{1-p}\,N^{-1}\left(\frac{d}{l \cdot m}\right) - N^{-1}(\bar{p})\right)\right)$$

$$\Leftrightarrow N^{-1}(\bar{p}) = \sqrt{1-p}\,N^{-1}\left(\frac{d}{l \cdot m}\right) - \sqrt{p}\,N^{-1}(\alpha)$$

So (5) then implies that

$$N^{-1}(\bar{p}) = \sqrt{1-0.19}\,N^{-1}\left(\frac{23}{0.6 \cdot 1000}\right) - \sqrt{0.19} \cdot N^{-1}(0.565)$$

$$\approx -1.6647$$

Hence, $N^{-1}(\bar{p}) = -1.6647 \quad (6)$

Recall that $m = 1000$

Now, using (6), and $p = 0.99$ and (8.4) Equation (3) in (1) where $C = a$ and $C = b$ we get that

$$F\left(\frac{a_0}{l \cdot m}\right) = F\left(\frac{40}{0.6 \cdot 1000}\right) \overset{(3)\,\text{etc.}}{\approx} 0.7641$$

$$F\left(\frac{b_0}{l \cdot m}\right) = F\left(\frac{90}{0.6 \cdot 1000}\right) \overset{(3)\,\text{etc.}}{\approx} 0.9534$$

and

$$P[a_0 < L_m < b_0] \approx 0.9534 - 0.7641 = 0.1893$$

Answer task 8: 18.93%

Let $N_m = \sum_{i=1}^{m} X_i$ where $p(Z) = P[X_i = 1 | Z]$ (1)

We want to compute $P[N_m = 0]$. (2)

We know that for any $k = 0, 1, \dots, m$ we have that

$$P[N_m = k] = E\left[\binom{m}{k} p(Z)^k (1 - p(Z))^{m-k}\right]. \quad (3)$$

and if Z is a discrete random variable taking the M different values $z_1, z_2, \dots, z_M$ then (3) can be rewritten as

$$P[N_m = k] = \sum_{j=1}^{M} \binom{m}{k} p(z_j)^k (1 - p(z_j))^{m-k} P[Z = z_j] \quad (4)$$

In our case $M = 2$, $z_1 = 1$, $z_2 = 2$ and $p(j)$ and $P[Z = j]$ are given by numbers in Table 1 in task 9.

9.2

Hence, if $k=0$ we can then rewrite (4) as

$$P[N_m=0]=\sum_{j=1}^{2}\binom{m}{0}p(j)^0(1-p(j))^{m-0}P[Z=j]$$

$$=\left\{\begin{array}{l}\binom{m}{0}=\frac{m!}{0!(m-0)!}=\frac{m!}{m!}=1\\ (1-p(j))^{m-0}=(1-p(j))^m,\ p(j)^0=1\end{array}\right\}=$$

$$=(1-p(1))^m P[Z=1]+(1-p(2))^m P[Z=2]$$

Hence,

$$P[N_m=0]=(1-p(1))^m P[Z=1]+(1-p(2))^m P[Z=2] \quad (5)$$

So with $m=25$ and $p(j)$, $P[Z=j]$ for $j=1,2$ given by Table 1, we finally get that $P[N_m=0]=0.4275$

Answer task 9: 42.75%

Task 10: Financial risk, 2018-08-24 (10.1)

In the mixed binomial logit-normal model we have that

$$P[X_i = 1 | Z] = p(Z) = \frac{1}{1+e^{-(\mu+\sigma Z)}} \quad (1)$$

where $Z \sim N(0,1)$, i.e. Z is a standard normal random variable.

We want to find μ and σ in (1) and then compute $VaR_L(\alpha)$ in this model for $\alpha = 95\%$ when calibrated to a Merton mixed binomial model according to

$$F_M(x_i) = F_{LogN}(x_i) \quad \text{for } i=1,2 \quad (2)$$

$$\text{and } x_1 = 0.1 \text{ and } x_2 = 0.9 \quad (3)$$

where $F_M(x) = N(a(x,\bar{p},\rho)) \quad (4)$

with $a(x,\bar{p},\rho) = \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right) \quad (5)$

Furthermore,

$$F_{LogN}(x) = P[p(Z) \leq x] \quad (6)$$

where $p(Z)$ is given by (1).

If $p(x) = \dfrac{1}{1+e^{-(\mu+\sigma x)}}$ then $p'(x)$ is

$$p'(x) = \frac{d}{dx}\left(\left(1+e^{-(\mu+\sigma x)}\right)^{-1}\right) =$$

$$= \underbrace{(-1)(-\sigma)e^{-(\mu+\sigma x)}}_{=\text{ inner derivate}} \left(1+e^{-(\mu+\sigma x)}\right)^{-2} =$$

$$= \frac{\sigma e^{-(\mu+\sigma x)}}{\left(1+e^{-(\mu+\sigma x)}\right)^2} > 0 \text{ for all } x$$

Hence, $p'(x) > 0$ for all x so $p(x)$ is strictly increasing and we therefore have that

$$F_{LogN}(x) = P[p(Z) \leq x] = P[Z \leq p^{-1}(x)]$$

$$= N(p^{-1}(x))$$

where $N(x)$ is the distribution func. to a stand. norm

10.3

Thus, $F_{\log N}(x) = N(p^{-1}(x))$ (6)

We next find $p^{-1}(x)$. by solving for x in the equation $y = p(x) \Leftrightarrow x = p^{-1}(y)$

So $p(x) = y \Leftrightarrow \dfrac{1}{1+e^{-(\mu+\sigma x)}} = y \Leftrightarrow$

$1+e^{-(\mu+\sigma x)} = \dfrac{1}{y} \Leftrightarrow e^{-(\mu+\sigma x)} = \dfrac{1-y}{y} \Leftrightarrow$

$x = -\dfrac{1}{\sigma}\left(\ln\left(\dfrac{1-y}{y}\right)+\mu\right)$

Hence, $p^{-1}(x) = \dfrac{1}{\sigma}\left(\ln\left(\dfrac{1-x}{x}\right)-\mu\right)$ (7)

So (2)-(5) and (6)-(7) implies that

$N(p^{-1}(x_i)) = N(a(x_i, \bar{p}, \rho))$ for $i=1,2$

$\Leftrightarrow p^{-1}(x_i) = a(x_i, \bar{p}, \rho)$ for $i=1,2$

so $\dfrac{1}{\sigma}\left(\ln\left(\dfrac{x_1}{1-x_1}\right)-\mu\right) = a(x_1, \bar{p}, \rho)$ (8)

$\dfrac{1}{\sigma}\left(\ln\left(\dfrac{x_2}{1-x_2}\right)-\mu\right) = a(x_2, \bar{p}, \rho)$ (9)

But (8) implies that

$$\mu = \ln\left(\frac{x_1}{1-x_1}\right) - \sigma \cdot a(x_1,\bar{p},p) \qquad (10)$$

and (10) in (9) then yields that

$$a(x_2,\bar{p},p) = \frac{1}{\sigma}\left(\ln\left(\frac{x_2}{1-x_1}\right) - \ln\left(\frac{x_1}{1-x_2}\right) + \sigma a(x_1,\bar{p},p)\right)$$

$\Leftrightarrow$

$$\sigma = \frac{\ln\left(\frac{x_2}{1-x_2}\right) - \ln\left(\frac{x_1}{1-x_1}\right)}{a(x_2,\bar{p},p) - a(x_1,\bar{p},p)} \qquad (11)$$

and (11) in (12) finally gives that

$$\mu = \ln\left(\frac{x_1}{1-x_1}\right) - a_1(x_1,\bar{p},p) \cdot \left(\frac{\ln\left(\frac{x_2}{1-x_2}\right) - \ln\left(\frac{x_1}{1-x_1}\right)}{a(x_2,\bar{p},p) - a(x_1,\bar{p},p)}\right) \qquad (12)$$

So with $\bar{p} = 3.5\%$, $p = 14\%$, $x_1 = 0.1$, $x_2 = 0.9$ then Equati. (11) & (12) gives that

$$\sigma = 0.6918 \qquad (13)$$

$$\mu = -3.3499 \qquad (14)$$

(10.5)

By definition we have that

$$\mathrm{VaR}_\alpha(L_m) = F_{L_m}^{\leftarrow}(\alpha) \quad (15)$$

where $F_{L_m}(x) = P[L_m \leq x]$ (16)

and $L_m = l \cdot N_m$ with $N_m = \sum_{i=1}^{m} X_i$

and $F_{L_m}^{\leftarrow}(x)$ is the generalized inverse to $F_{L_m}(x)$. From the LPA-theory we know that

$$F_{L_m}(x) = P[L_m \leq x] = P\left[\frac{N_m}{m} \leq \frac{x}{l \cdot m}\right] \to F\left(\frac{x}{l \cdot m}\right) \quad (17)$$

when $m \to \infty$ and where $F(x) = P[p(Z) \leq x]$

In the logit-normal case we have that

$$F(x) = F_{\mathrm{logN}}(x) = P[p(Z) \leq x] \overset{(6)}{=} N(p^{-1}(x))$$

that is $F(x) = N(p^{-1}(x))$ (18)

So (17) implies that

$$F_{L_m}(x) \approx F\left(\frac{x}{l \cdot m}\right) \quad (19)$$

and (19) implies that

$$F^{\leftarrow}_{L_m}(x) \approx l \cdot m \cdot F^{\leftarrow}(x) \quad (20)$$

and in the logit-normal case, $F(x)$ is continuous so $F^{\leftarrow}(x) = F^{-1}(x)$.
and (20) can be rewritten as

$$F^{-1}_{L_m}(x) \approx l \cdot m \, F^{-1}(x).$$

Thus, $VaR_\alpha(L) \approx l \cdot m \cdot F^{-1}(x)$ (21)
when $F(x)$ is continuous

To find $F^{-1}(x)$ we solve for x in the equation $y = F(x) \Leftrightarrow x = F^{-1}(y)$
Thus, in the logit normal case we have that

$$y = F(x) \Leftrightarrow y = N(p^{-1}(x)) \Leftrightarrow x = p(N^{-1}(y))$$

Hence, $F^{-1}(x) = p(N^{-1}(x))$ (22)

10.7

Thus, (21) & (22) then implies that in the logit-normal case we have that (in the LPA case)

$$VaR_\alpha(L) \approx l \cdot m \cdot p(N^{-1}(\alpha))$$

so from (1) we finally get that

$$VaR_\alpha(L) \approx \frac{\tilde{l} \cdot m}{1+e^{-(\mu+\sigma N^{-1}(\alpha))}} \qquad (23)$$

Thus, with $m = 1000$, $l = 0.6$, $\alpha = 0.95$ and σ, μ given by (13) & (14) we ~~get~~ with these numbers that

$$VaR_{95\%}(L) \approx 59.21 \text{ millions } \$$$

where we used linearity of $VaR_\alpha(L)$ and that each loan has notional of $1 \cdot 10^6$

Answer task 10: 59.21 millions US-dollar