GAUSSIAN COPULA

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Innehållsförteckning

INTRODUCTION

The aim of this report is to give the reader an exploratory introduction of the Gaussian Copula, and how it was utilized to asses credit risk and to price credit derivatives, e.g. **Collateralized Debt Obligations (CDO’s).** Additionally, the report will also consist of a discussion about the role the Gaussian Copula possessed during the Financial Crisis in 2008, and how it amplified the severity of the crisis.

The report will be structured into two parts: Firstly, a background and a mathematical description of CDO's and the Gaussian Copula will be presented. Secondly, a discussion about the model's abilities will be held and an analysis of why the Gaussian Copula failed during the financial crisis will be done.

BACKGROUND

COLLATERALIZED DEBT OBLIGATIONS

A Collateralized Debt Obligation is a financial instrument which bundles together several different cash flow-generating assets (mostly debt assets) into bonds, which later on can be sold to investors. A CDO is essentially a package of debt assets, such as auto loans, mortgages or corporate debt (Dexner & Zerihoum, 2018). The process of creating CDO's starts off with transforming these bundles of debt into financial derivatives that can be sold and traded, and the transformation is feasible through a process named securitization. The process of securitization makes it possible to transfer this pool or bundle of debt from the banks, which are often the owners of the bundles of debt assets, to so-called "Special Purpose Vehicles" (SPV’s). An SPV is virtually a trust or a tax-excused company with the sole purpose to redirect the cash flow generated by the underlying debt assets in the form of issued bonds to the investors who have purchased the CDO. More specifically, the SPV will use the cash generated by the underlying assets to pay the purchaser and holder of the CDO in the form of coupons (Oxford, 2006).

Furthermore, a CDO is typically sliced into different risk categories referred to as tranches. There exist three different risk categories: Senior, Mezzanine and Equity. The Senior tranche possesses the least risk, but it receives the smallest coupon payment. The Mezzanine tranche is exposed to a medium risk level; which is compensated through a larger coupon payment. Lastly, the Equity tranche is the riskiest tranche as well as the tranche that generates the highest yield for its investors. The principle behind the tranches is based on the link between the risk of default on the underlying debt assets and in which order each tranche obtains its coupon. In the case of default in the pool of assets, the tranche with least seniority (Equity tranche) will run the risk of not receiving its coupon payment. Thus, the Senior tranche will be the first tranche in the CDO to receive its coupon payment, and consequently will be the most protected tranche in the case of default (Oxford, 2006). This principle is illustrated in Figure #1.

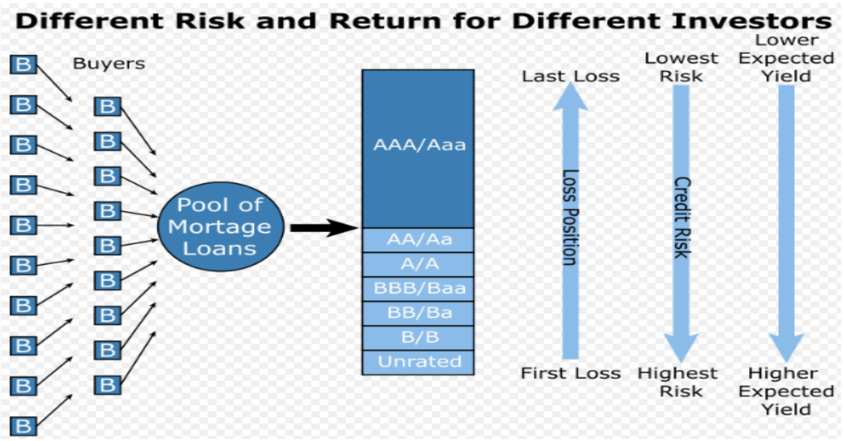


Figure #1

THE GAUSSIAN COPULA APPROACH

During the late 1990s, the business of trading CDO's gained increasingly more traction and popularity due to its lucrative nature. Despite the immense development, there existed a considerable issue with the CDO's: how to price them accordingly. David X. Li, a partner at J.P. Morgan at the time, presented a solution to the pricing issue. Li argued that a Gaussian Copula model would suffice and would be appropriate to use when pricing CDO's and assessing the credit risk which builds up these lucrative financial instruments (Li, 2000).

A Copula is a multivariate probability distribution where each variable is uniformly marginally distributed. The power that comes with the copula function is such that it allows the user to decompose the estimation of a multidimensional distribution of some random variables into an estimation of the individual marginal distributions and the joint dependency structure between the random variables at matter (Meucci, 2011). Essentially, a copula is a tool that models dependence between several random variables and the term was first introduced into probability theory by Abe Sklar in 1959. Sklar's theorem is stated in [1]. In the Gaussian case, we use the multivariate normal distribution to describe the dependence structure between some random variables. The formula for the Gaussian Copula is given by [2].

The way Li applied the Gaussian Copula approach to model default correlation started off by introducing a random variable T_i which represented the **survival time** of some asset *i.* For two assets, say A and B, the default correlation between them is given by

$$\rho_{AB} = \frac{Cov(T_A, T_B)}{\sqrt{Var(T_A)Var(T_B)}} = \frac{E(T_A T_B) - E(T_A)E(T_B)}{\sqrt{Var(T_A)Var(T_B)}}$$

and is stated as **the survival time correlation** in Li's infamous paper "On Default Correlation: A Copula Function Approach". Furthermore, Li argued that the most advantageous way to model the survival time distribution is to use the **hazard function.** Using the hazard function, which is stated in [3], one must choose a method to obtain the term structure of default rates. One such method is to use historical data on default rates from rating agencies; but contrary to this proposition Li suggested using an implied approach instead, which was based on market prices of default bonds or asset swap spreads. This is due to several reasons, and one of them is that market prices rapidly adapt to changes in the market and another one is that most rating agencies only consider default frequency in their data, rather of taking default severity into account (Li, 2000). Concludingly, the Gaussian Copula function is applied and the formula for the joint default probability for credit A and B follows as

$$P[T_A < 1, T_B < 1] = \Phi_2(\Phi^{-1}\big(F_A(1)\big), \Phi^{-1}(F_B(1), \rho)$$

where F_A and F_B are the distribution functions for the survival times T_A and T_B, Φ_2 is the bivariate accumulative normal distribution function.

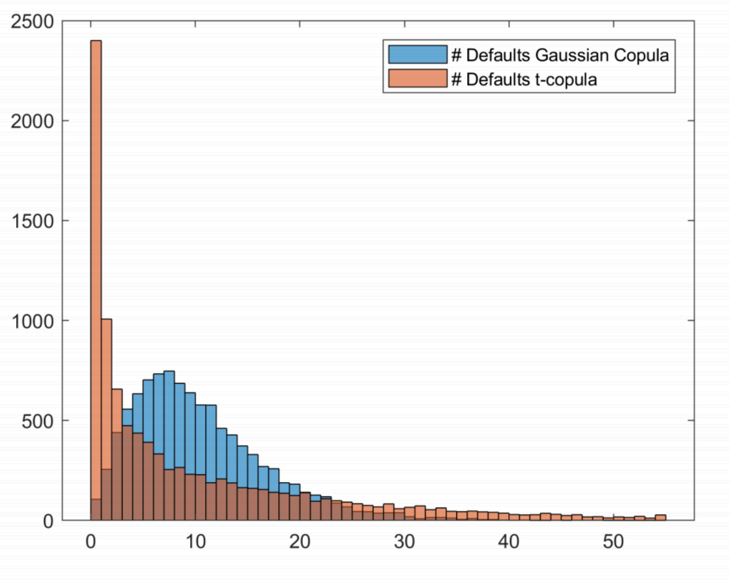
LIMITATIONS OF THE GAUSSIAN COPULA

Li’s approach to the correlation problem was a breakthrough at the time of its introduction, mostly because it was considered a more sophisticated and easier approach than it´s precursors. The adoption of the Gaussian copula was therefore followed by significant growth in the market of credit derivatives. This growth, however, became the foundation on which the biggest financial crisis of the modern era was built upon. Some issues with this rapid growth were that the corporate CDO world relied almost exclusively on the Gaussian copula, in other words, the people involved in the evaluation of these credit derivatives relied heavily on mathematical models. This model, like all models, is built on assumptions and simplifications. These mathematical models used to model the correlation between defaults were complex, and therefore hard to comprehend. Naturally, managers and many others had a hard time understanding the model and they became easy to misuse and misinterpret (Salmon, 2009). During the fast growth of the credit derivative market, the banks dismissed warnings, partly due to lack of understanding, partly because of having an exaggerated risk-taking attitude. The banks were simply making too much money to stop.

In the aftermath of the financial crisis, journalists and business analysts suggested that one of the main sources of the crisis was Li’s formula. The criticism given to the Gaussian copula points to a specific property of the Gaussian copula: tail-independence. Tail independence means that in case of extreme events, both high and low the correlation vanishes. When assuming tail independence regarding default, it follows that defaults appear with low correlation (Meucci, 2011). This means that for instance, only a small portion of a portfolio defaults now and then – but never a large portion at the same time. This assumption, however, has been historically confuted, where the US subprime mortgage crisis may be the best example. Neglecting tail-dependence in the models essentially means ignoring the risk of extreme events; events that can cause investors to lose all of their investments.

Furthermore, in the summer of 2007, it became evident that the US subprime market was in turmoil and homeowners across the US started to default on their mortgagees in masses. As soon as the banks started to experience these huge losses regarding the CDO market, global liquidity shrivelled, and banks stopped lending each other money due to the fear of insolvency. Li’s formula and approach did not foresee such events, due to the nature of the formula. The Gaussian approach assumed that events gravitated around a mean value, i.e. an "ordinary" state. When it comes to mortgages and loans, there exists an extremely large set of possible outcomes and the realm of mortgages and loans is far from binary. Markets, such as the mortgage market, experienced far more extreme correlation events than what the Gaussian Copula approach could ever model and capture.

There exists a number of different copulas which differ strongly in the dependence structure of the tails and therefore, the choice of a copula will have a significant impact on the distribution of the losses. An illustration of the effects of the copula is shown in figure #. The model was made in MATLAB, the illustration conducted a portfolio of N =100 obligors who were given a common default probability of p = 0.1. The interdependence of the defaults is modelled with an N-dimensional copula. Random samples were drawn, and the experiment was repeated M = 10 000 times. A histogram was then created from the number of defaults:



PERSONAL CONCLUSIONS

As shown, the loss distribution depends heavily on the choice of a copula. The Gaussian copula disregards extreme events and might not be an optimal approach when it comes to modelling complex and random financial markets and their interrelatedness. Other copulas that exhibit tail-dependence would have been more suitable choices.

There is an old saying in statistics: "All models are wrong, but some are useful". The cheap debt made available by the finance revolution resulted in many investors looking to put their money in debt, disregarding the limits of Li's approach and relying far too heavily on the mathematical models used to price the extensive supply of financial instruments available. Li’s model was useful but also flawed. A more cautious approach would have been preferable. In the same spirit as the Taleb’s Turkey example, we conclude: “If one does not understand the real-world situation well enough, the best quantitative tools will not help”.

Appendix

[1] **Sklar's Theorem**
Let H be a two-dimensional distribution function with marginal distribution F and G. Them there exists a copula C such that

$$H(x,y) = C\big(F(x), G(y)\big)$$

Conversely, for any univariate distribution functions F and G and any copula C, the function H is a two-dimensional distribution function with marginals F and G. Furthermore, if G and F are continuous, then C is unique.

[2] **Gaussian Copula function:**

$$C_R^G\big(u_{1,} \dots, u_d\big) = \Phi_R\big(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\big).$$

Where R is the correlation matrix and $\Phi^{-1}(\cdot)$ *is the inverse of the standard normal cumulative distribution function.*

[3] **The Hazard function**
The hazard function h(x) is the ratio of the probability density function P(x) to the survival function S(x), given by

$$h(x) = \frac{P(x)}{S(x)} = \frac{P(x)}{1 - D(x)},$$

where D(x) is the distribution function.

Further Reading

#1 In the background section of this report was a rather brief discussion about CDO's and how this financial instrument is created. For a deeper insight about this instrument, the following article will provide the reader with a comprehensive explanation:

Dexner, Karl-Erik and Zerihoum, Johan. "Collateralized Debt Obligations – A Study on the Informational Transaction Transparency", (Fall 2018), *School of Business, Economics and Law Göteborg University,* Available at: https://gupea.ub.gu.se/bitstream/2077/19598/1/gupea_2077_19598_1.pdf

#2 An intuitive and exploratory guide in the world of copulas, where the article decomposes the complex mathematical models and concepts in a comprehensive fashion:

Meucci, Attilio. "A Short, Comprehensive, Practical Guide to Copulas", (October 2011), *Global Association of Risk Professionals,* The Quant Classrom ,Available at: http://www.garp.org/media/691726/quant_classroom_oct2011.pdf

#3 The original paper by David X. Li which introduced the Gaussian copula approach to the financial markets and initiated a series of events examined in our report:

Li, David X. “On Default Correlation: A Copula Function Approach”, (April 2000), *The Riskmetrics Group,* Working Paper Number 99-07, Available at: http://www.maths.lth.se/matstat/kurser/fmsn15masm23/default.pdf

#4 Finally, an article that captures and explains the whole story about the events discussed in our report in a fun and entertaining way:

Salmon, Felix. “Recipe for disaster: The formula that killed Wall Street”*,* (February 23, 2009), *Wired,* Available at: https://www.wired.com/2009/02/wp-quant/

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Salmon, Felix. “Recipe for disaster: The formula that killed Wall Street”, (February 23, 2009), *Wired,* Available at: https://www.wired.com/2009/02/wp-quant/