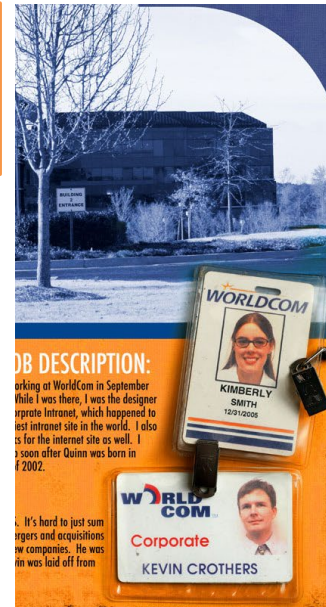




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Financial Risk  
2-rd quarter 2012/2013  
Tuesdays 10:15-12  
Thursdays 13:15-15:00  
in MVF31 and Pascal

Financial Risk  
4-rd quarter  
2018/19



"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."



Gudrun January 2005  
326 MEuro loss  
72 % due to forest losses  
4 times larger than second largest 3



Gudrun January 2005  
326 MEuro loss  
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"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

# Maximum Likelihood (ML) inference (Coles p. 30-43)

Likelihood function = the function which shows how the “probability” (or likelihood) of getting the observed data depends on the parameters

$x_1, \dots, x_n$  observations of i.i.d. variables  $X_1, \dots, X_n$ , density  $f(x) = f(x; \theta)$

$\theta = (\theta_1, \dots, \theta_d)$  parameters

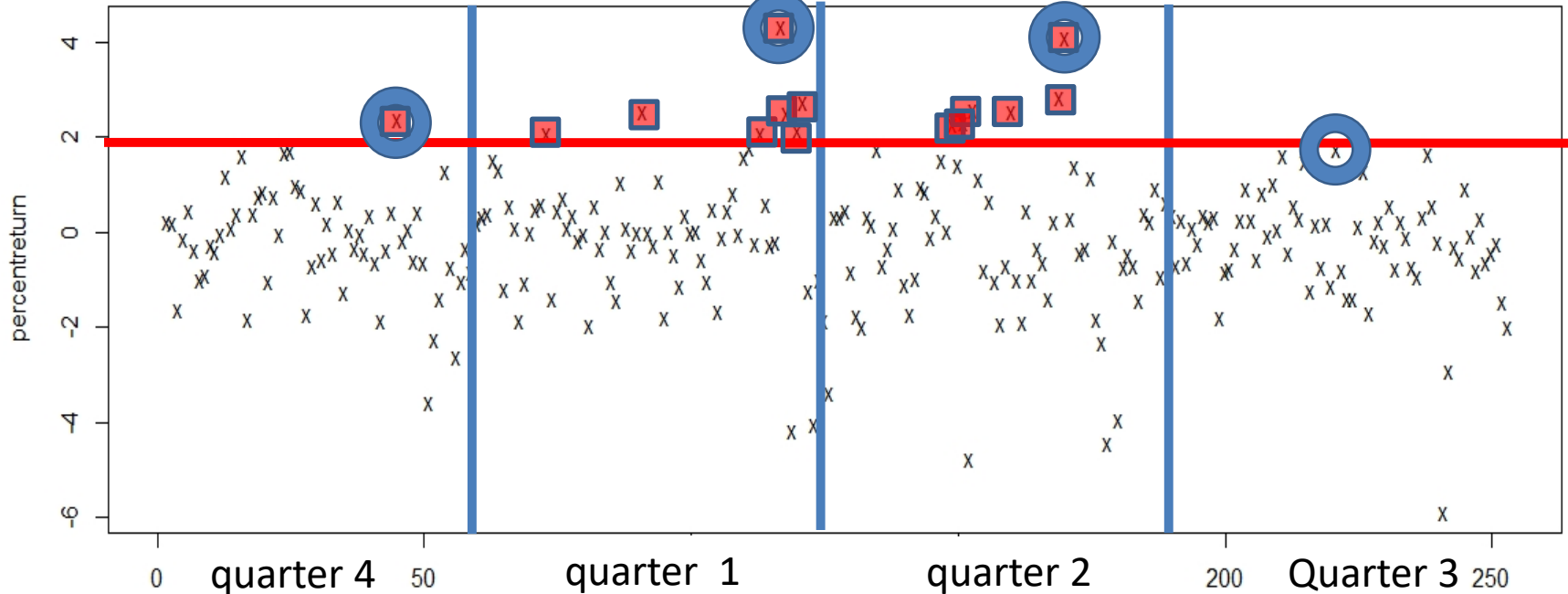
$L(\theta) = f(x_1; \theta)f(x_2; \theta) \dots f(x_n; \theta)$  likelihood function

$\ell(\theta) = \log f(x_1; \theta) + \log f(x_2; \theta) + \dots \log f(x_n; \theta)$  log likelihood function

ML estimates = the value  $\hat{\theta} = (\hat{\theta}_1 \dots \hat{\theta}_d)$  which maximizes the (log) likelihood function

- ML estimates often have to be found through numerical maximization
- sometimes a maximum doesn't exist
- sometimes several local maxima ( $\rightarrow$  problem for numerical maximization)
- but typically no problems if the number of observations is “large”

Apple losses ( $= -100 \times \frac{\text{price tomorrow} - \text{price today}}{\text{price today}}$ ) one year back



○ Maximum quarterly loss     
 ■ excess of the level  $u = 1.92$

**How large is the risk of a big quarterly loss? BM**

**How large is the risk of a big loss tomorrow? PoT**

**Example:** ML estimation of the parameters in the PoT model

$T$  = length of observation period

$N$  = number of observed excesses (random variable!)

$x_1, \dots, x_N$  observed excess sizes

$\theta = (\sigma, \gamma, \lambda)$  parameters

The probability of observing  $N$  excesses is  $\frac{(\lambda T)^N}{N!} \exp\{-\lambda T\}$   
independence  $\Rightarrow$

$$L(\theta) = L(\lambda, \sigma, \gamma) = \frac{(\lambda T)^N}{N!} \exp\{-\lambda T\} \prod_{i=1}^N \frac{1}{\sigma} \left(1 + \frac{\gamma}{\sigma} x_i\right)_+^{-1/\gamma-1}$$

$$\begin{aligned} \ell(\lambda, \sigma, \gamma) &= N \log(\lambda) + N \log(T) - \log(N!) - \lambda T \\ &\quad - N \log(\sigma) - \sum_{i=1}^N (1/\gamma + 1) \log \left(1 + \frac{\gamma}{\sigma} x_i\right)_+ \end{aligned}$$

$$\frac{\partial}{\partial \lambda} \ell(\lambda, \sigma, \gamma) = \frac{N}{\lambda} - T = 0 \quad \text{so that} \quad \hat{\lambda} = \frac{N}{T}$$

$\hat{\sigma}, \hat{\gamma}$  obtained from numerical maximization of the second part of  $\ell(\lambda, \sigma, \gamma)$

## ML inference: asymptotic properties

$\mathcal{I}(\theta) = E_{\theta} \left( - \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell(\theta) \right)$  expected Fisher information matrix, estimated by  $\mathcal{I}(\hat{\theta})$  or by  $I(\theta)$  where  $I(\theta) = \left( - \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell(\theta) \right)$  is the the observed Fisher information matrix. (In the expected Fisher information matrix, the observations are replaced by the corresponding random variables when the expectations are computed. Numerical optimization programs typically compute the Hessian,  $-I(\theta)$  )

$\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_d)$  asymptotically has a d-dimensional multivariate normal distribution with mean  $\theta$  and variance  $\mathcal{I}(\theta)^{-1}$

In particular, the variance of  $\hat{\theta}_i$  may be estimated by  $(\mathcal{I}(\hat{\theta})^{-1})_{ii}$  = the  $i$ -th diagonal element of  $\mathcal{I}(\hat{\theta})^{-1}$ , or by  $(I(\hat{\theta})^{-1})_{ii}$ . The latter is often more accurate.

$k_{\alpha}$  = the  $1-(1-\alpha)/2$ -th quantile of the standard normal distribution

$(\hat{\theta}_i - k_{\alpha} \sqrt{(I(\hat{\theta})^{-1})_{i,i}}, \hat{\theta}_i + k_{\alpha} \sqrt{(I(\hat{\theta})^{-1})_{i,i}})$  asymptotic  $100\alpha\%$   
confidence interval

**Exercise:** Compute a confidence interval for the parameters in a Poisson process

$t$  = length of observation period = 5 years

$n$  = number of observed excesses = 31

$\lambda$  = parameter (= intensity) of Poisson process

**Solution:**

$$\ell(\lambda) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$\log(\ell(\lambda)) = n \log(\lambda) + n \log(t) - \log(n!) - \lambda t$$

$$\frac{d \log(\ell(\lambda))}{d\lambda} = \frac{n}{\lambda} - t = 0 \Rightarrow \hat{\lambda} = \frac{n}{t} = \frac{31}{5} = 6.25$$

$$I(\lambda) = -\frac{d^2 \log(\ell(\lambda))}{d\lambda^2} = \frac{n}{\lambda^2}$$

Estimated variance of  $\hat{\lambda}$  is  $-\frac{1}{I(\hat{\lambda})} = \frac{\hat{\lambda}^2}{n}$

95% confidence interval:

$$\begin{aligned} & (\hat{\lambda} - k_{0.95} \sqrt{\hat{\lambda}^2/n}, \hat{\lambda} + k_{0.95} \sqrt{\hat{\lambda}^2/n}) \\ & = (6.25 - 1.96 \frac{6.25}{\sqrt{31}}, 6.25 + 1.96 \frac{6.25}{\sqrt{31}}) \end{aligned}$$

## ML inference: the delta method

$\eta = g(\theta) = g(\theta_1, \dots, \theta_d)$  function of the parameters

$\hat{\eta} = g(\hat{\theta}) = g(\hat{\theta}_1, \dots, \hat{\theta}_d)$  estimate of the function of the parameters

$\nabla(\theta) = (\frac{\partial}{\partial \theta_1} g(\theta), \dots, \frac{\partial}{\partial \theta_d} g(\theta))$  gradient,  $\nabla(\hat{\theta})$  estimate of gradient

$\hat{\eta}$  asymptotically normal with mean  $\eta$  and variance  $\nabla(\theta) \mathcal{I}(\theta)^{-1} \nabla(\theta)^t$   
(which e.g. can be estimated by  $\nabla(\hat{\theta}) I(\hat{\theta})^{-1} \nabla(\hat{\theta})^t$ ).

From this one can construct confidence intervals for  $\eta$  in the same way as the confidence intervals for  $\theta$  on the previous page.

Works well if  $g$  is approximately linear, not so well otherwise



**Exercise:** Compute a confidence interval for the 95% quantile in a GP distribution based on observations  $x_1, x_2, \dots, x_N$

$$\text{Density } h(x; \sigma, \gamma) = \frac{1}{\sigma} \left(1 + \frac{\gamma}{\sigma} x\right)^{-\frac{1}{\gamma} - 1}$$

$$\log h(x; \sigma, \gamma) = -\log \sigma - \left(\frac{1}{\gamma} + 1\right) \log \left(1 + \frac{\gamma}{\sigma} x\right)$$

$$\frac{d}{d\sigma} \log h(x; \sigma, \gamma) = -\frac{1}{\sigma} + \frac{1 + \gamma}{\sigma^2} \frac{1}{\left(1 + \frac{\gamma}{\sigma} x\right)}$$

$$\frac{d}{d\gamma} \log h(x; \sigma, \gamma) = \frac{1}{\gamma^2} \log \left(1 + \frac{\gamma}{\sigma} x\right) - \left(\frac{1}{\gamma} + 1\right) \frac{1}{\sigma} \frac{1}{\left(1 + \frac{\gamma}{\sigma} x\right)}$$

log likelihood function

$$\ell(\sigma, \gamma) = \sum_{i=1}^N \log h(x_i; \sigma, \gamma)$$

ML-estimates  $\hat{\sigma}, \hat{\gamma}$  obtained as solutions to the equations

$$\sum_{i=1}^N \frac{d}{d\sigma} \log h(x_i; \sigma, \gamma) = 0$$

$$\sum_{i=1}^N \frac{d}{d\gamma} \log h(x_i; \sigma, \gamma) = 0$$

Observed information matrix

$$I(\sigma, \gamma) = \begin{pmatrix} -\frac{d^2}{d\sigma^2} \ell(\sigma, \gamma) & -\frac{d^2}{d\sigma d\gamma} \ell(\sigma, \gamma) \\ -\frac{d^2}{d\sigma d\gamma} \ell(\sigma, \gamma) & -\frac{d^2}{d\gamma^2} \ell(\sigma, \gamma) \end{pmatrix}$$

The 95% quantile  $x_{.95}$  in a GP distribution is obtained by solving

$$H(x_{.95}) = 1 - \left(1 + \frac{\gamma}{\sigma} x\right)^{-\frac{1}{\gamma}} = 0.95$$

The solution is

$$g(\sigma, \gamma) = x_{.95} = \frac{\sigma}{\gamma} (0.05^{-\gamma} - 1)$$

so the maximum likelihood estimate is

$$\widehat{x}_{.95} = \frac{\widehat{\sigma}}{\widehat{\gamma}} (0.05^{-\widehat{\gamma}} - 1)$$

$$\begin{aligned} \nabla(\sigma, \gamma) &= \left( \frac{d}{d\sigma} g(\sigma, \gamma), \frac{d}{d\gamma} g(\sigma, \gamma) \right) \\ &= \left( \frac{1}{\gamma} (0.05^{-\gamma} - 1), -\frac{\sigma}{\gamma^2} (0.05^{-\gamma} - 1) - \frac{\gamma}{\sigma} (-\log 0.05) 0.05^{-\gamma} \right) \end{aligned}$$

The standard error of  $\widehat{x}_{.95}$  is estimated as

$$\text{SE}(\widehat{x}_{.95}) = \sqrt{\nabla(\widehat{\sigma}, \widehat{\gamma}) \ I(\widehat{\sigma}, \widehat{\gamma})^{-1} \nabla(\widehat{\sigma}, \widehat{\gamma})^T}$$

and a 95% asymptotic confidence interval is

$$(\widehat{x}_{.95} - 1.96\text{SE}(\widehat{x}_{.95}), \widehat{x}_{.95} + 1.96\text{SE}(\widehat{x}_{.95}))$$

## ML inference: Likelihood Ratio (LR) tests

$\theta = (\theta_1, \theta_2)$  partition of  $\theta$  into two vectors  $\theta_1$  and  $\theta_2$  of dimensions  $d - p$  and  $p$ .  $\hat{\theta}_2^*$  maximizes  $l(\theta_1, \theta_2)$  over  $\theta_2$ , for  $\theta_1$  “kept fixed” (so function of  $\theta_1$ )

$2(\ell(\hat{\theta}) - \ell(\theta_1, \hat{\theta}_2^*))$  asymptotically has a  $\chi^2$  distribution with  $d - p$  degrees of freedom if  $\theta_1$  is the true value  $\rightarrow$  LR test:

Reject  $H_0 : \theta_1 = \theta_1^0$  at the significance level  $\alpha\%$  if

$2 \left( l(\hat{\theta}) - l(\theta_1^0, \hat{\theta}_2^*) \right) > \chi_\alpha^2(d - p)$ , where  $\chi_\alpha^2(d - p)$  is the  $\alpha$ -th quantile of the  $\chi^2$  distribution with  $d - p$  degrees of freedom

# ML inference: profile likelihood confidence intervals

(often more accurate than delta method intervals, *plots from Coles* )

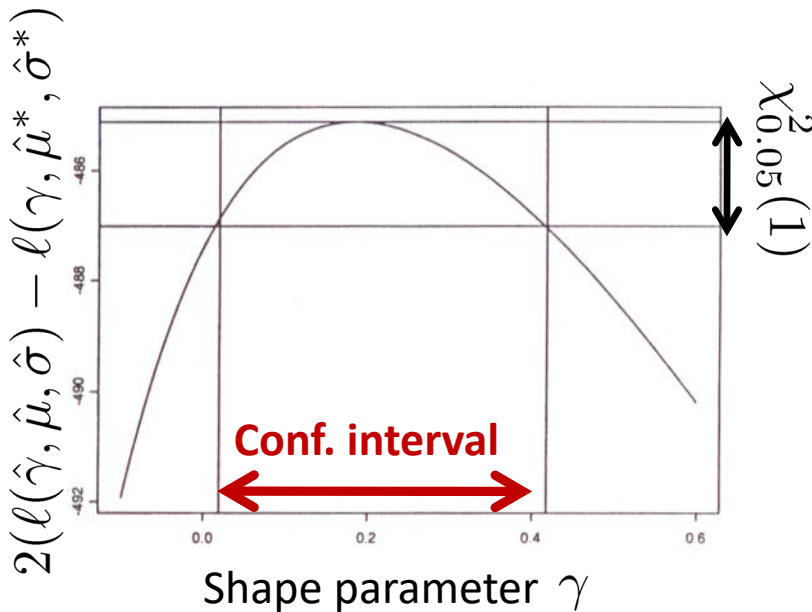


FIGURE 4.3. Profile likelihood for  $\xi$  in threshold excess model of daily rainfall data.

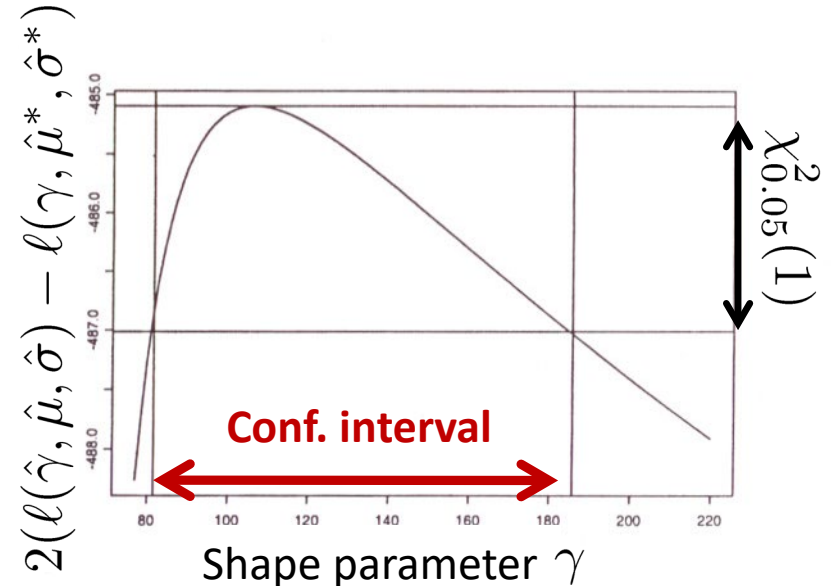


FIGURE 4.4. Profile likelihood for 100-year return level in threshold excess model of daily rainfall data.

Profile likelihood confidence intervals for the shape parameter in the Block Maxima model. The delta method would give similar interval in the left case, but not in the right.

$x_p = VaR_p(L) = p$ -th quantile of distribution of loss  $L$   
= solution to  $F_L(x_p) = p$

$ES_p(L) = E(L|L > VaR_p(L)) =$  Expected Shortfall

For the PoT model with threshold  $u$  suppose that  $P(L > u) = p_u$ . Then

$$VaR_p(L) = \frac{\sigma}{\gamma} \left\{ \left( \frac{1-p}{p_u} \right)^{-\gamma} - 1 \right\},$$

provided this value is greater than  $u$ , and

$$ES_p(L) = VaR_p(L) + \frac{\sigma + \gamma(VaR_p(L) - u)}{1 - \gamma}$$

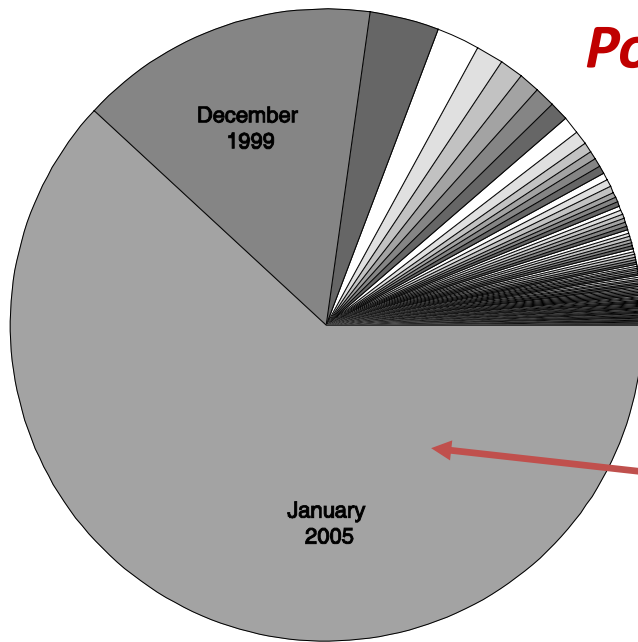
**Important exercise:** Check if the formulas on the previous page are correct.

$VaR_p(L)$  and  $ES_p(L)$  are estimated by replacing  $\sigma, \gamma$  in the formulas on the previous page by their estimates  $\hat{\sigma}, \hat{\gamma}$  and replacing  $p_u$  by its estimate

$$p_u^* = \frac{\# \text{ excesses of } u}{\# \text{ observations}}$$



**PoT: windstorm insurance (Rootzén & Tajvidi)**



Windstorm losses for  
Länsförsäkringar 1982-2005

Gudrun January 2005

326 MEuro loss

72 % due to forest losses

4 times larger than second largest



**The real problem!**

# The problems

How much reinsurance should LFAB buy?

Should LFAB worry about windstorm losses getting worse?

How should LFAB adjust if its forest insurance portfolio grows?

and:

Can detailed modeling give better risk estimates?

Are windstorms becoming more frequent?

**1994 PoT analysis of 1982-1993 LFAB data** (the basic method, more sophisticated analysis of 1982-2005 data in later paper)

Risk (MSEK)	next year	next 5 years	next 15 years
10%	66	215	473
1%	366	1149	2497

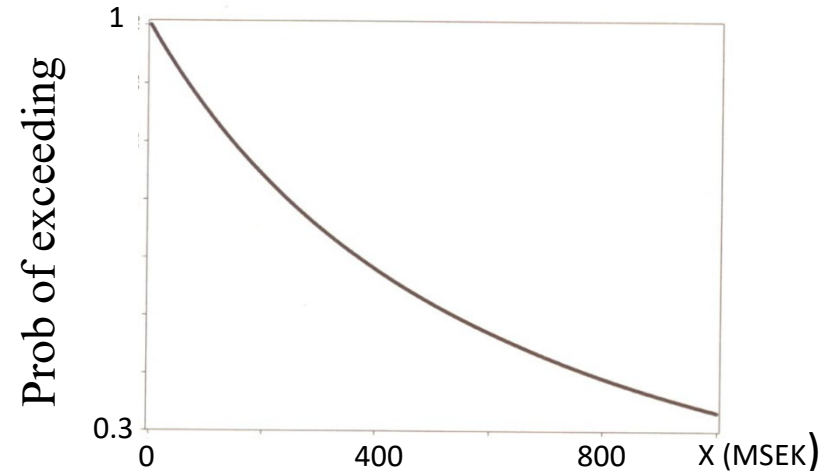
$$X_i \sim GP(y; \sigma_t, \gamma)$$

$$\sigma_t = \exp(\alpha + \beta t)$$

$$\hat{\alpha} = 0.93$$

$$\hat{\beta} = .013 \pm .013$$

no evidence of trend in extremes



conditional probability that a loss in excess of the reinsurance level 850 MSEK exceeds x

Gudrun: 2912 MSEK, 12 years later

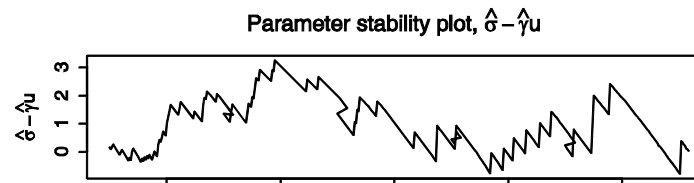
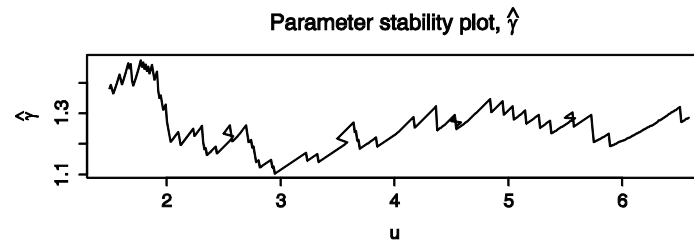
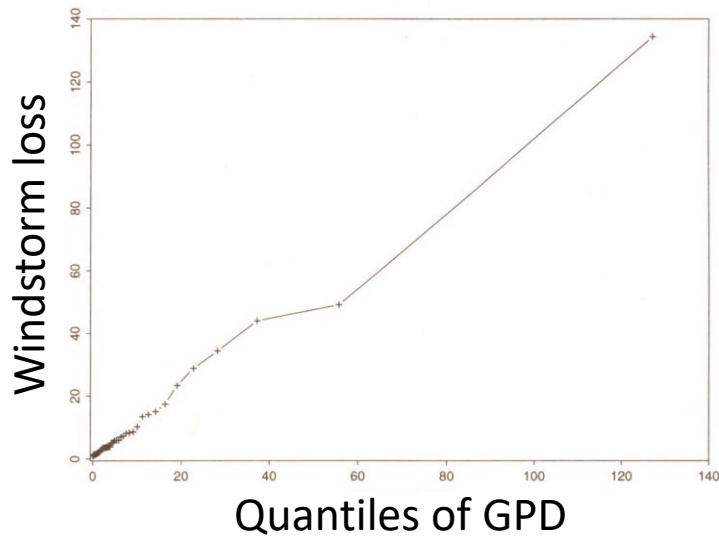
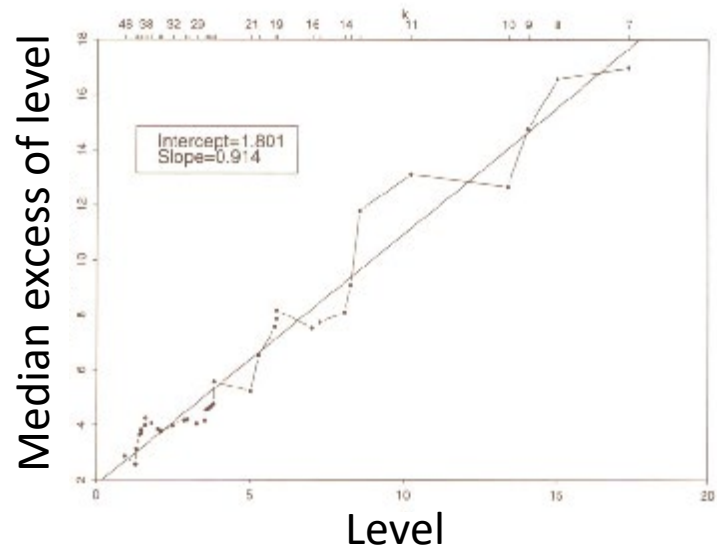
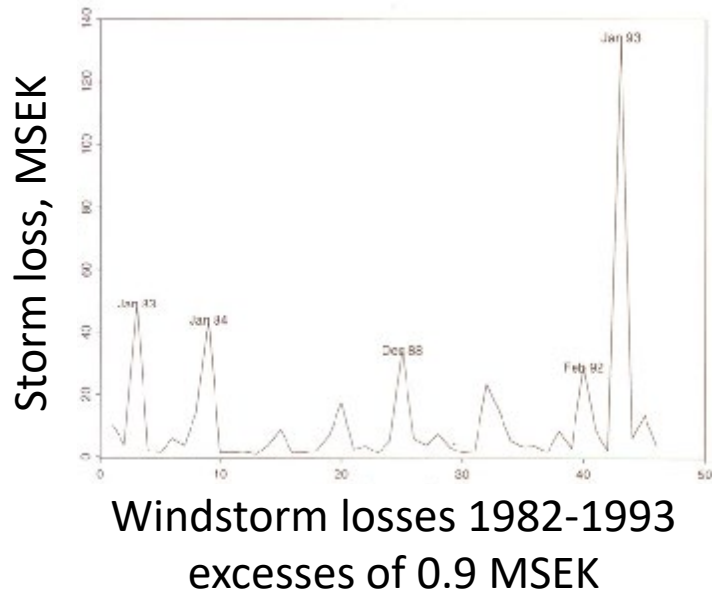
Windstorms of 1902 and 1969 probably comparable to Gudrun

## Choice of threshold/number of order statistics in PoT, model diagnostics

Threshold choice compromise between low bias (= good fit of model): requires high threshold/few order statistics, and low variance: requires low threshold/many order statistics

- mean excess plots (high variability for heavy tails)
- median excess plots
- plots of parameter estimates as function of threshold/number of order statistics
- qq- and pp-plots

automatic threshold selection procedures exist, and are getting better, but still “optimal” threshold depends on the unknown underlying distribution which has to be estimated.



Windstorm losses 1982-2005  
parameter stability plots

## Some conclusions

- risk cannot be summarized into one number
- extreme value statistics provide the simplest methods (but other methods may sometimes be needed)
- didn't find clear trends
- meteorological data didn't help
- don't trust computer simulation models unless statistically validated
- companies should develop systematic techniques for thinking about “not yet seen” catastrophes
- put contractual limits to aggregate exposure

# A step in another direction: catastrophe risks

**BIG --- "happens only once"**

- can't adjust and improve as experience is gained
- methods based on means, variances, central limit theory have little meaning
- difficult to keep in mind that catastrophes can (and will!) occur

**a gamble --- find the odds of a gamble!**

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