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EX1 $m=1000 \quad \bar{p}=5\% \quad l=60\% \quad \varrho=25\%$ Each loan 1M SEK

$$P(L_m \le 100) = P(L_m \le 100) - P(L_m \le 50)$$

$$\approx N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{100}{l\cdot m}\right) - N^{-1}(\bar{p})\right)\right) - N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{50}{l\cdot m}\right) - N^{-1}(\bar{p})\right)\right)$$

$$= 0.1323$$

EX2 $m=1000 \quad \bar{p} \quad l=0.60 \quad \varrho=0.12$ Each loan 1M SEK

$P(L_m \le 20) = 0.535$

$P(50 \le L_m \le 100) = 0.9925 - 0.9047$

$= 0.0878.$

$$P(L_m \le x) \approx N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{lm}\right) - N^{-1}(\bar{p})\right)\right)$$

$$N^{-1}(P(L_m \le x)) = \frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{lm}\right) - N^{-1}(\bar{p})\right)$$

$$N^{-1}(\bar{p}) = \sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{lm}\right) - \sqrt{\varrho}\, N^{-1}(P(L_m \le x))$$

$$= -1.7508$$

EX3 $m=1000 \quad l=0.60$ logit-normal mixing

LPA-VaR formula

$$\boxed{\mathrm{VaR}_\alpha(L) = l\cdot m \cdot F^{\leftarrow}(\alpha)}$$

$$F(x) = N(p^{-1}(x)) \Rightarrow F^{\leftarrow}(\alpha) = p(N^{-1}(\alpha))$$

$$= \frac{1}{1+\exp(-(\mu+\sigma N^{-1}(\alpha)))}$$

$$\sigma = \frac{\ln\left(\frac{1}{1-\mathrm{var}95/l\cdot m} - 1\right) - \ln\left(\frac{1}{1-\mathrm{var}99/l\cdot m} - 1\right)}{N^{-1}(0.95) - N^{-1}(0.99)} = 1.6305$$

$$\mu = \ln\left(\frac{1}{1-\mathrm{VaR}95/lm} - 1\right) - \sigma N^{-1}(0.95) = -4.1008$$

$\mathrm{VaR}_{99.9\%}(L) = 431.1885. \rightarrow \boxed{431 \text{ M SEK}}$

Ex 4 m obligors Mixed binomial model, $p(Z) = \frac{Z}{K}$, $Z \sim Bin(K, q)$

$K \in \mathbb{N}$, $K \geq 2$. $m = 10$, $K = 3$, $q = 0.05$.

Compute $P(N_m = 0)$.

$$P(N_m = k) = E\left[\binom{m}{k} p(Z)^k (1-p(Z))^{m-k}\right] \qquad (2.65)$$

so

$$P(N_m = 0) = E\left[\binom{m}{0} p(Z)^0 (1-p(Z))^m\right]$$

$$= E\left[(1-p(Z))^m\right]$$

$$= \sum_{i=0}^{K} (1-p(i))^m \cdot P(Z=i)$$

$$= \sum_{i=0}^{K} \left(1-\frac{i}{K}\right)^m \binom{K}{i} q^i (1-q)^{K-i}$$

$$= 0.859 \quad \boxed{\Rightarrow 86\%}$$

Ex 5 m obligors. Mixed binomial model, $p(Z) = \frac{Z}{K}$, $Z \sim Bin(K, q)$

$m = 100$, $K = 3$, $q = 0.05$

Compute $\rho_X = Corr(X_i, X_j)$ for $i \neq j$ (Default dependence / correlation)

$$\rho_X = \frac{E[p(Z)^2] - \bar{p}^2}{\bar{p}(1-\bar{p})}, \quad \bar{p} = E[p(Z)]$$

$$E[p(Z)] = \sum_{z=0}^{K} p(z) \cdot P(Z=z) = \sum_{z=0}^{K} \frac{z}{K} \cdot \binom{K}{z} q^z (1-q)^{K-z}$$

$$= 0.0500$$

$$E[p(Z)^2] = \sum_{z=0}^{K} \left(\frac{z}{K}\right)^2 \binom{K}{z} q^z (1-q)^{K-z} = 0.0120$$

$$\rho_X = 0.2$$

<u>Ex 6</u> m obligors, mixed beta binomial model. $p(Z) = Z$, $Z \sim Beta(a,b)$

$a = 2$, $E[p(Z)] = 0.1$. Compute $\rho_X = Corr(X_i, X_j)$, $i \neq j$.

$$\rho_X = \frac{E[p(Z)^2] - E[p(Z)]^2}{E[p(Z)](1 - E[p(Z)])} = \frac{E[Z^2] - E[Z]^2}{E[Z](1 - E[Z])} = \frac{Var(Z)}{E[Z](1 - E[Z])}.$$

$$E[Z] = \frac{a}{a+b} \qquad E[Z^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$Var(Z) = \frac{ab}{(a+b)^2(a+b+1)}.$$

So

$$\rho_X = \frac{ab/(a+b)^2(a+b+1)}{a/(a+b) \cdot (1 - a/(a+b))} = \frac{\frac{ab}{(a+b)^2(a+b+1)}}{\frac{a}{a+b} \cdot \frac{b}{a+b}} = \frac{1}{a+b+1}$$

$0.1 = E[p(Z)] = E[Z] = \frac{a}{a+b}$ and $a = 2$

so $\frac{2}{2+b} = 0.1$, i.e. $\frac{2 - 0.2}{0.1} = b = 18$

and thus

$$\rho_X = \frac{1}{2+18+1} = \frac{1}{21} = 0.0476.$$

Ex 7 m obligors, mixed binomial model with lognormal mixing dist

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EX20 $\quad F(x) = N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}(x) - N^{-1}(\bar{p})\right)\right)$

where

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \int_{-\infty}^{x} f(u)\, du.$$

so

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$\frac{d}{dx} N^{-1}(x) = \frac{1}{N'(N^{-1}(x))} = \frac{1}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(N^{-1}(x))^2}{2}\right)} = \sqrt{2\pi}\, e^{(N^{-1}(x))^2/2}$$

Therefore,

$$\frac{dF(x)}{dx} = N'\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}(x) - N^{-1}(\bar{p})\right)\right) \cdot \frac{\sqrt{1-\varrho}}{\sqrt{\varrho}} \left(\frac{d}{dx} N^{-1}(x)\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}(x) - N^{-1}(\bar{p})\right)\right)^2}{2}\right) \frac{\sqrt{1-\varrho}}{\sqrt{\varrho}} \sqrt{2\pi}\, e^{(N^{-1}(x))^2/2}$$

$$= \frac{\sqrt{1-\varrho}}{\sqrt{\varrho}} \exp\left(\frac{1}{2}(N^{-1}(x))^2 - \frac{1}{2\varrho}\left(N^{-1}(\bar{p}) - \sqrt{1-\varrho}\, N^{-1}(x)\right)^2\right).$$

EX16 $\quad P(L_m \le x) = N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{l \cdot m}\right) - N^{-1}(\bar{p})\right)\right)$

To find $VaR_\alpha(L) = F_L^{\leftarrow}(\alpha)$, let $P(L_m \le x) = \alpha$ and solve for x.

$$\alpha = N\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{lm}\right) - N^{-1}(\bar{p})\right)\right)$$

$$\sqrt{\varrho}\, N^{-1}(\alpha) = \sqrt{1-\varrho}\, N^{-1}\left(\frac{x}{lm}\right) - N^{-1}(\bar{p})$$

$$N^{-1}\left(\frac{x}{lm}\right) = \frac{\sqrt{\varrho}\, N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\varrho}}$$

$$VaR_\alpha(L) = x = l \cdot m \cdot N\left(\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} N^{-1}(\alpha) + \frac{N^{-1}(\bar{p})}{\sqrt{1-\varrho}}\right).$$

EX9

$$\bar{p} = P(X_i = 1) = P(X_i = 1 \mid Z = 1) P(Z=1) + P(X_i = 1 \mid Z=2) P(Z=2)$$

$$= 0.03 \cdot 0.7 + 0.12 \cdot 0.3 = 0.057$$

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Var}(p(Z)) = E(p(Z)^2] - E[p(Z)]^2$$

$$= 0.00495 - 0.057^2 = 0.0017$$

$$\uparrow$$

$$E[p(Z)^2] = p(1)^2 P(Z=1) + p(2)^2 P(Z=2) = 0.03^2 \cdot 0.7 + 0.12^2 \cdot 0.3$$

$$= 0.00495$$

So

$$\rho_X = \frac{\mathrm{Var}(p(Z))}{\bar{p}(1-\bar{p})} = 0.0316 .$$

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EX 13 $m=1000$. Mixed binomial model w/ logit-normal mixing distribution

$\mu = -2.5371$, $l = 0.60$, $VaR_{95\%}(L) = 172.3$ M SEK

$P(30 \le L_m \le 80) = ?$

$VaR_\alpha(L) \simeq l \cdot m \cdot F^{\leftarrow}(\alpha)$

$F(x) = P(p(Z) \le x) \approx P(\frac{N_m}{m} \le x)$

$F(x) = N(p^{-1}(x))$

$F^{-1}(\alpha) = p(N^{-1}(\alpha)) = \frac{1}{1+\exp(-(\mu+\sigma N^{-1}(\alpha)))}$

First we need σ:

$$p^{-1}(x) = \frac{1}{\sigma}\left(\ln\left(\frac{x}{1-x}\right) - \mu\right)$$

$$\Rightarrow \sigma = \frac{\ln(x/(1-x)) - \mu}{p^{-1}(x)}$$

$$\overset{(*)}{=} \frac{\ln\left(\frac{VaR_{95\%}(L)}{l \cdot m}\right) - \ln\left(1 - \frac{VaR_{95\%}(L)}{l \cdot m}\right) - \mu}{N^{-1}(0.95)}$$

$$= 0.9897$$

Assume
$VaR_\alpha(L) = l \cdot m \cdot F^{-1}(\alpha)$

Then
$F^{-1}(\alpha) = \frac{VaR_\alpha(L)}{l \cdot m}$

i.e.

$$\underbrace{p^{-1}(F^{-1}(\alpha))}_{= N^{-1}(\alpha)} = p^{-1}\left(\frac{VaR_\alpha(L)}{l \cdot m}\right) \quad (*)$$

Now

$$P(L_m \le x) = P\left(\frac{N_m}{m} \le \frac{x}{l \cdot m}\right) \approx N\left(p^{-1}\left(\frac{x}{l \cdot m}\right)\right)$$

So

$$P(30 \le L_m \le 80) = P(L_m \le 80) - P(L_m \le 30) = 0.7493 - 0.3403 = \underline{\underline{0.4090}}$$

EX 15

$$F_{L_m}(x) = \sum_{k=0}^{\lfloor \frac{x}{l} \rfloor} P(N_m = k) = \sum_{k=0}^{\lfloor \frac{x}{l} \rfloor} E[P(N_m = k \mid Z)]$$

$\frac{x}{l} = \frac{1.1}{0.6} = 1.83$ so

$$P(L_m \le 1.1) = F_{L_m}(1.1) = \sum_{k=0}^{1} E[P(N_m = k \mid Z)] = \sum_{k=0}^{1}\sum_{j=1}^{2} \binom{m}{k} p(j)^k (1-p(j))^{m-k} P(Z=j)$$

$$= \binom{m}{0} p(1)^0 (1-p(1))^m P(Z=1) + \binom{m}{0} p(2)^0 (1-p(2))^m P(Z=2)$$

$$+ \binom{m}{1} p(1)^1 (1-p(1))^{m-1} P(Z=1) + \binom{m}{1} p(2)^1 (1-p(2))^{m-1} P(Z=2)$$

$$= (1-0.03)^{20} \cdot 0.7 + (1-0.12)^{20} \cdot 0.3 + 20 \cdot 0.03 \cdot (1-0.03)^{19} + 20 \cdot 0.12 (1-0.12)^{19} 0.3$$

$$= 0.7028$$

EX21

$$ES_\alpha(L) = E[L \mid L \geq VaR_\alpha(L)]$$

$$= \frac{E[L \cdot 1_{\{L \geq VaR_\alpha(L)\}}]}{P(L \geq VaR_\alpha(L))}$$

$$= \frac{1}{1-\alpha} E[L \cdot 1_{\{L \geq VaR_\alpha(L)\}}]$$

$$= \frac{1}{1-\alpha} \int_{VaR_\alpha(L)}^{\infty} x \cdot f(x)\,dx$$

$$\overset{(*)}{\overset{I}{=}} \frac{1}{1-\alpha} \int_\alpha^1 F_L^{-1}(u) \cdot du$$

$$= \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)\,du$$

$$\boxed{\begin{array}{l} VaR_\alpha(L) = F_L^{-1}(\alpha) \\ x = F_L^{-1}(u) \\ u = F_L(x) \qquad (*) \\ du = f(x)\,dx \end{array}}$$