Lecture 4

Last updated by Serik Sagitov: November 22, 2012

Abstract

A course based on the book Probabilities and Random Processes by Geoffrey Grimmett and David Stirzaker. Chapter 6. Markov chains (6.4-6.5, 6.8-6.9).

1 Stationary distributions

A vector of probabilities $\boldsymbol{\pi} = (\pi_j, j \in S)$ is a stationary distribution for the Markov chain X_n, if given X_0 has distribution $\boldsymbol{\pi}$, X_n has the same distribution $\boldsymbol{\pi}$ for any n, or in other words $\boldsymbol{\pi}$ is a left eigenvector of the transition matrix

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}.$$

Theorem 1 *An irreducible chain (aperiodic or periodic) has a stationary distribution $\boldsymbol{\pi}$ iff the chain is positively recurrent; in this case $\boldsymbol{\pi}$ is the unique stationary distribution and is given by $\pi_i = 1/\mu_i$, where $\mu_i = \mathbb{E}(T_i|X_0 = i)$ and T_i is the time of first return to i.*

Proof*. Let $\boldsymbol{\rho}(k) = (\rho_j(k), j \in S)$ where $\rho_k(k) = 1$ and

$$\rho_j(k) = \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, T_k \geq n | X_0 = k)$$

is the mean number of visits of the chain to the state j between two consecutive visits to state k. Then

$$\begin{aligned}\sum_{j\in S} \rho_j(k) &= \sum_{j\in S}\sum_{n=1}^{\infty} \mathbb{P}(X_n = j, T_k \geq n | X_0 = k) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(T_k \geq n | X_0 = k) = \mathbb{E}(T_k | X_0 = k) = \mu_k.\end{aligned}$$

Main steps of the proof:

1. if the chain is irreducible recurrent, then $\rho_j(k) < \infty$ for any k and j, and furthermore, $\boldsymbol{\rho}(k)\mathbf{P} = \boldsymbol{\rho}(k)$,
2. if the chain is irreducible recurrent, there exists a positive root $\mathbf{x}$ of the equation $\mathbf{x}\mathbf{P} = \mathbf{x}$, which is unique up to a multiplicative constant; the chain is positively recurrent iff $\sum_{j\in S} x_j < \infty$.

Theorem 2 *Let s be any state of an irreducible chain. The chain is transient iff there exists a non-zero bounded solution $(y_j : j \neq s)$ satisfying $|y_j| \leq 1$ for all j to the equations*

$$y_i = \sum_{j\in S\backslash\{s\}} p_{ij} y_j, \quad i \in S\backslash\{s\}. \tag{1}$$

Proof*. Main step. Let τ_j be the probability of no visit to s ever for a chain started at state j. Then the vector $(\tau_j : j \neq s)$ satisfies (1).

Example 3 Random walk with retaining barrier. Transition probabilities

$$p_{00} = q, \quad p_{i-1,i} = p, \quad p_{i,i-1} = q, \quad i \geq 1.$$

Let $\rho = p/q$.

- If $q < p$, take $s = 0$ to see that $y_j = 1 - \rho^{-j}$ satisfies (1). The chain is transient.
- Solve the equation $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ to find that there exists a stationary distribution, with $\pi_j = \rho^j(1-\rho)$, if and only if $q > p$.
- If $q > p$, the chain is positively recurrent, and if $q = p = 1/2$, the chain is null recurrent.

Theorem 4 *Ergodic theorem. For an irreducible aperiodic chain we have that*

$$p_{ij}^{(n)} \to \frac{1}{\mu_j} \text{ as } n \to \infty \text{ for all } (i,j).$$

More generally, for an aperiodic state j and any state i we have that $p_{ij}^{(n)} \to \frac{f_{ij}}{\mu_j}$, where f_{ij} is the probability that the chain ever visits j starting at i.

2 Reversibility

Theorem 5 *Put $Y_n = X_{N-n}$ for $0 \le n \le N$ where X_n is a stationary Markov chain. Then Y_n is a Markov chain with*

$$\mathbb{P}(Y_{n+1} = j | Y_n = i) = \frac{\pi_j p_{ji}}{\pi_i}.$$

The chain Y_n is called the time-reversal of X_n. If $\boldsymbol{\pi}$ exists and $\frac{\pi_j p_{ji}}{\pi_i} = p_{ij}$, the chain X_n is called reversible (in equilibrium). The detailed balance equations

$$\pi_i p_{ij} = \pi_j p_{ji} \text{ for all } (i,j). \tag{2}$$

Theorem 6 *Consider an irreducible chain and suppose there exists a distribution $\boldsymbol{\pi}$ such that (2) holds. Then $\boldsymbol{\pi}$ is a stationary distribution of the chain. Furthermore, the chain is reversible.*

Proof. Using (2) we obtain

$$\sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji} = \pi_j.$$

Example. Ehrenfest model of diffusion: flow of m particles between two connected chambers. Pick a particle at random and move it to another chamber. Let X_n be the number of particles in the first chamber. State space $S = \{0, 1, \ldots, m\}$ and transition probabilities

$$p_{i,i+1} = \frac{m-i}{m}, \qquad p_{i,i-1} = \frac{i}{m}.$$

The detailed balance equations

$$\pi_i \frac{m-i}{m} = \pi_{i+1} \frac{i+1}{m}$$

imply

$$\pi_i = \frac{m-i+1}{i} \pi_{i-1} = \binom{m}{i} \pi_0.$$

Using $\sum_i \pi_i = 1$ we find that the stationary distribution $\pi_i = \binom{m}{i} 2^{-n}$ is a symmetric binomial.

3 Poisson process and continuous-time Markov chains

A Poisson process $N(t)$ with intensity λ is the number of events observed up to time t given that the inter-arrival times are independent exponentials with parameter λ:

$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k \ge 0.$$

More generally, a pure birth process $X(t)$ with intensities $\{\lambda_i\}_{i=0}^{\infty}$

- holds at state i an exponential time with parameter λ_i,
- after the holding time it jumps up from i to $i+1$.

Exponential holding times has no memory and therefore imply the Markov property in the continuous time setting.

Explosion: $\mathbb{P}(X(t)=\infty)>0$ for a finite t. It is possible iff $\sum 1/\lambda_i<\infty$.

Definition 7 *A continuous-time process $X(t)$ with a countable state space S satisfies the Markov property if*

$$\mathbb{P}(X(t_n)=j|X(t_1)=i_1,\ldots,X(t_{n-1})=i_{n-1})=\mathbb{P}(X(t_n)=j|X(t_{n-1})=i_{n-1})$$

for any states $j,i_1,\ldots,i_{n-1}\in S$ and any times $t_1<\ldots<t_n$.

In the time homogeneous case compared to the discrete time case instead of transition matrices $\mathbf{P}^n$ with elements $p_{ij}^{(n)}$ we have transition matrices $\mathbf{P}_t$ with elements

$$p_{ij}(t)=\mathbb{P}(X(u+t)=j|X(u)=i).$$

Chapman-Kolmogorov: $\mathbf{P}_{t+s}=\mathbf{P}_t\mathbf{P}_s$ for all $t\geq 0$ and $s\geq 0$. Here $\mathbf{P}_0=\mathbf{I}$ is the identity matrix.

Example. For the Poisson process we have

$$p_{ij}(t)=\frac{(\lambda t)^{j-i}}{(j-i)!}e^{-\lambda t}$$

and

$$\sum_k p_{ik}(t)p_{kj}(s)=\sum_k\frac{(\lambda t)^{k-i}}{(k-i)!}e^{-\lambda t}\frac{(\lambda s)^{j-k}}{(j-k)!}e^{-\lambda s}=\frac{(\lambda t+\lambda s)^{j-i}}{(j-i)!}e^{-\lambda(t+s)}=p_{ij}(t+s).$$

4 The generator of a continuous-time Markov chain

A generator $\mathbf{G}=(g_{ij})$ is a matrix with non-negative off-diagonal elements such that $\sum_j g_{ij}=0$. A Markov chain $X(t)$ with generator $\mathbf{G}$

- holds at state i an exponential time with parameter $\lambda_i=-g_{ii}$,
- after the holding time it jumps from i to $j\neq i$ with probability $h_{ij}=\frac{g_{ij}}{\lambda_i}$.

The embedded discrete Markov chain is governed by transition matrix $\mathbf{H}=(h_{ij})$ satisfying $h_{ii}=0$. A continuous-time MC is a discrete MC plus holding intensities (λ_i).

Example. The Poisson process and birth process have the same embedded MC with $h_{i,i+1}=1$. For the birth process $g_{ii}=-\lambda_i$, $g_{i,i+1}=\lambda_i$ and all other $g_{ij}=0$.

Kolmogorov equations. Forward equation: for any $i,j\in S$

$$p'_{ij}(t)=\sum_k p_{ik}(t)g_{kj}$$

or in the matrix form $\mathbf{P}'_t=\mathbf{P}_t\mathbf{G}$. It is obtained from $\mathbf{P}_{t+\epsilon}-\mathbf{P}_t=\mathbf{P}_t(\mathbf{P}_\epsilon-\mathbf{P}_0)$ watching for the last change. Backward equation $\mathbf{P}'_t=\mathbf{G}\mathbf{P}_t$ is obtained from $\mathbf{P}_{t+\epsilon}-\mathbf{P}_t=(\mathbf{P}_\epsilon-\mathbf{P}_0)\mathbf{P}_t$ watching for the initial change. These equations often have a unique solution

$$\mathbf{P}_t=e^{t\mathbf{G}}:=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{G}^n.$$

Theorem 8 *Stationary distribution: $\boldsymbol{\pi}\mathbf{P}_t=\boldsymbol{\pi}$ for all t iff (a counterpart of the discrete time equation $\boldsymbol{\pi}\mathbf{P}=\boldsymbol{\pi}$)*

$$\boldsymbol{\pi}\mathbf{G}=\mathbf{0}.$$

Proof:

$$\boldsymbol{\pi}\mathbf{P}_t \overset{\forall t}{=} \boldsymbol{\pi} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} \frac{t^n}{n!}\boldsymbol{\pi}\mathbf{G}^n \overset{\forall t}{=} \boldsymbol{\pi} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{t^n}{n!}\boldsymbol{\pi}\mathbf{G}^n \overset{\forall t}{=} \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{\pi}\mathbf{G}^n \overset{\forall n}{=} \mathbf{0}.$$

Example. Check that the birth process has no stationary distribution.

Theorem 9 *Let $X(t)$ be irreducible with generator $\mathbf{G}$. If there exists a stationary distribution $\boldsymbol{\pi}$, then it is unique and for all (i, j)*

$$p_{ij}(t) \to \pi_j, \quad t \to \infty.$$

If there is no stationary distribution, then $p_{ij}(t) \to 0$ as $t \to \infty$.

Example. Poisson process holding times $\lambda_i = \lambda$. Then $\mathbf{G} = \lambda(\mathbf{H} - \mathbf{I})$ and $\mathbf{P}_t = e^{\lambda t(\mathbf{H}-\mathbf{I})}$.