Lecture 5

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Abstract

A course based on the book Probabilities and Random Processes by Geoffrey Grimmett and David Stirzaker. Chapter 7.1-7.3. Convergence of random variables.

1 Modes of convergence

Theorem 1 *If $X_1, X_2, \ldots$ are random variables defined on the same probability space, then so are*

$$\inf_n X_n, \quad \sup_n X_n, \quad \liminf_n X_n, \quad \limsup_n X_n.$$

Proof. For any real x

$$\{\omega : \inf_n X_n \le x\} = \bigcup_n \{\omega : X_n \le x\} \in \mathcal{F}, \quad \{\omega : \sup_n X_n \le x\} = \bigcap_n \{\omega : X_n \le x\} \in \mathcal{F}.$$

It remains to observe that

$$\liminf_n X_n = \sup_n \inf_{m \ge n} X_m, \qquad \limsup_n X_n = \inf_n \sup_{m \ge n} X_m.$$

Definition 2 *Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define four modes of convergence of random variables*

- *almost sure convergence $X_n \stackrel{\text{a.s.}}{\to} X$, if $\mathbb{P}(\omega : \lim X_n = X) = 1$,*
- *convergence in r-th mean $X_n \stackrel{L^r}{\to} X$ for a given $r \ge 1$, if $\mathbb{E}|X_n^r| < \infty$ for all n and*

$$\mathbb{E}(|X_n - X|^r) \to 0,$$

- *convergence in probability $X_n \stackrel{\text{P}}{\to} X$, if for all positive ϵ*

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \tag{1}$$

- *convergence in distribution $X_n \stackrel{d}{\to} X$ (does not require a common probability space), if*

$$\mathbb{P}(X_n \le x) \to \mathbb{P}(X \le x) \text{ for all } x \text{ such that } \mathbb{P}(X = x) = 0.$$

Theorem 3 *Let $q \ge r \ge 1$ and c be a constant. The following implications hold*

$$X_n \stackrel{L^q}{\to} X \Rightarrow \begin{array}{c} X_n \stackrel{\text{a.s.}}{\to} X \Rightarrow \\ X_n \stackrel{L^r}{\to} X \Rightarrow \end{array} \quad X_n \stackrel{\text{P}}{\to} X \Rightarrow \quad X_n \stackrel{d}{\to} X,$$

$$X_n \stackrel{d}{\to} c \Rightarrow X_n \stackrel{\text{P}}{\to} c.$$

Moreover, under additional conditions

$$X_n \stackrel{\text{P}}{\to} X \Rightarrow \begin{array}{c} X_n \stackrel{L^r}{\to} X \\ X_n \stackrel{\text{a.s.}}{\to} X \end{array}.$$

Here for the convergence in mean it is required that $\mathbb{P}(|X_n| \le a) = 1$ for all n and some positive constant a. For the almost sure convergence it is required that (1) is strengthened as

$$\sum_n \mathbb{P}(|X_n - X| > \epsilon) < \infty \textit{ for any } \epsilon > 0. \tag{2}$$

It follows that

$$X_n \stackrel{\mathrm{P}}{\to} X \Rightarrow X_{n'} \stackrel{\mathrm{a.s.}}{\to} X \textit{ along a subsequence.}$$

The implication "convergence in mean $\Rightarrow$ convergence in probability" follows from the Markov inequality:

$$\mathbb{P}(|X| > \epsilon) \le \frac{\mathbb{E}(|X|)}{\epsilon} \text{ for any } \epsilon > 0.$$

The implication "(2) $\Rightarrow$ convergence a.s." follows from the first Borel-Cantelli lemma. Indeed, put $A_n = \{|X_n - X| > 1/m\}$ and $B_m = \{A_n \text{ i.o.}\}$. Observe that $\{\omega : \lim X_n \ne X\} = \sup_m B_m$ and due to the Borel-Cantelli lemma $\mathbb{P}(B_m) = 0$.

2 Borel-Cantelli lemmas

Given a sequence of random events $A_1, A_2, \ldots$ define new events

$$\sup_n A_n = \bigcup_n A_n, \qquad \inf_n A_n = \bigcap_n A_n,$$

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

Observe that

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\forall n\ \exists m \ge n \text{ such that } A_m \text{ occurs}\} = \{\text{events } A_m \text{ occur infinitely often}\}$$

and

$$A^c = \liminf_{n\to\infty} A_n^c = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c = \{\exists n \text{ such that } A_m^c \text{ occur } \forall m \ge n\} = \{\text{events } A_m \text{ occur finitely often}\}.$$

Theorem 4 *Borel-Cantelli lemmas. Let $A = \{$infinitely many of random events $A_1, A_2, \ldots$ occur$\}$.*

1. *If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A) = 0$,*
2. *If $\sum_{n=1}^{\infty} P(A_n) = \infty$ and events $A_1, A_2, \ldots$ are independent, then $P(A) = 1$.*

3 Counterexamples

Let $\Omega = [0, 2]$ and put $A_n = [a_n, a_{n+1}]$, where a_n is the fractional part of $1 + 1/2 + \ldots + 1/n$.

- The random variables I_{A_n} converge to zero in mean (and therefore in probability) but not a.s.
- The random variables nI_{A_n} converge to zero in probability but not in mean and not a.s.
- $\mathbb{P}(A_n \text{ i.o.}) = 0.5$ and $\sum_n \mathbb{P}(A_n) = \infty$.

Let $\Omega = [0, 1]$ and put $B_n = [0, 1/n]$.

- The random variables nI_{B_n} converge to zero a.s but not in mean.
- Put $X_{2n} = I_{B_2}$ and $X_{2n+1} = 1 - X_{2n}$. The random variables X_n converge to I_{B_2} in distribution but not in probability. The random variables X_n converge in distribution to $1 - I_{B_2}$ as well.
- Both $X_{2n} = I_{B_2}$ and $X_{2n+1} = 1 - X_{2n}$ converge to I_{B_2} in distribution but their sum does not converge to $2I_{B_2}$.

4 Continuity of expectation

Lemma 5 *(Fatou) If almost surely $X_n \geq 0$, then*

$$\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n) \leq \limsup \mathbb{E}(X_n) \leq \mathbb{E}(\limsup X_n).$$

In particular, applying this to $X_n = I_{\{A_n\}}$ we get

$$\mathbb{P}(\liminf A_n) \leq \liminf \mathbb{P}(A_n) \leq \limsup \mathbb{P}(A_n) \leq \mathbb{P}(\limsup A_n).$$

Theorem 6 *Dominated convergence. If $X_n \stackrel{\text{a.s.}}{\to} X$ and almost surely $|X_n| \leq Y$ and $\mathbb{E}Y < \infty$, then $\mathbb{E}|X| < \infty$ and $\mathbb{E}X_n \to \mathbb{E}X$.*

Bounded convergence theorem: if almost surely $|X_n| \leq c$ for some constant c.