Tentamentsskrivning i MVE 251, 7.5 hp.

Tid:
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Hjälpmedel: Chalmersgodkänd räknare, utdelad formelsamling.

A typical exam consists of 6 questions each worth of 5 points. Below come a number of examples of exam questions.

1. (5 points) Suppose that $\{X_n\}_{n=0}^N$ is an irreducible non-null persistent Markov chain, with transition matrix $\mathbf{P} = (p_{ij})$ and stationary distribution $\pi = (\pi_1, \pi_2, \ldots)$. Defined another chain $\{Y_n\}_{n=0}^N$ by $Y_n = X_{N-n}$.
   (a) Check that $\{Y_n\}_{n=0}^N$ is also a Markov chain and find its transition probabilities.
   (b) Under which additional condition the two chains $\{X_n\}_{n=0}^N$ and $\{Y_n\}_{n=0}^N$ have the same transition probabilities?

2. (5 points) Consider three independent Bernoulli variables $X_i \sim Be(p_i)$, $i = 1, 2, 3$.
   (a) Find the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$.
   (b) Compute the covariance between Y_1 and Y_2.
   (c) When Y_1 and Y_2 are independent?

3. (5 points) Given a sequence of random events $A_1, A_2, \ldots$ define a new event
$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$
   (a) What is the meaning of the event A? Explain in detail.
   (b) Show that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A) = 0$.

4. (5 points) Given a sequence of random events $A_1, A_2, \ldots$ define a new event
$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$
   (a) Suppose $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is such a sequence of sigma-fields that $A_n \in \mathcal{F}_n$ for all n. Show that A belongs to the corresponding tail sigma-field.
   (b) It is known that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A) = 0$. Can we combine this fact with the previous to conclude using a zero-one law that if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A) = 1$?

5. (5 points) Let S_n be a symmetric random walk with jumps ± 1 and starting from $S_0 = 0$. Define the maximum process by $M_n = \max\{S_0, S_1, \ldots, S_n\}$.
   (a) Draw an example of a trajectory $\{S_0, S_1, \ldots, S_n\}$ and the corresponding trajectory $\{M_0, M_1, \ldots, M_n\}$. Compare on this trajectory the values of S_n and M_n.
   (b) Show that
$$P(S_{10000} > 196) \approx 0.025,$$
   and
$$P(M_{10000} > 500) \le 0.02.$$

6. (5 points) Let $\{X_n\}_{n=1}^{\infty}$ represent the sequence of outcomes of independent tosses of a fair coin, so that $X_n = -1$ if n-th toss results in tails and $X_n = 1$ otherwise.

(a) Draw a sequence of Venn diagrams to illustrate the idea of a natural filtration $\{\mathcal{F}_n\}_{n=1}^{\infty}$ with $\mathcal{F}_n$ being the sigma-field generated by $\{X_1, \ldots, X_n\}$.

(b) Compute the conditional variance $\mathrm{Var}(X_1 + X_2 + X_3 | X_1 + X_2)$.

7. (5 points) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of stochastic variables defined by

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n),$$

where I_1, I_2, ϕ are independent random variables such that $P(I_1 = 1) = P(I_1 = -1) = P(I_2 = 1) = P(I_2 = -1) = \frac{1}{2}$ and ϕ is uniformly distributed over $[-\pi, \pi]$.

(a) On the same graph draw three examples of trajectories of the process $\{X_n\}$. What is random in these trajectories?

(b) Verify that $\{X_n\}$ is a stationary process.

(c) Find the spectral density function of $\{X_n\}$.

8. (5 points) An urn contains n balls numbered $1, 2, \ldots, n$. All balls are drawn one by one at random without replacement. Denote their numbers $X_1, \ldots, X_n$.

(a) Compute the conditional expectation $E(X_2 | X_1)$.

(b) Find the joint distribution of $(X_1, \ldots, X_n)$ and the marginal distributions.

9. (5 points) Weak convergence of random variables $X_n \xrightarrow{d} X$, if

$$\mathbb{P}(X_n \leq x) \to \mathbb{P}(X \leq x) \text{ for all } x \text{ such that } \mathbb{P}(X = x) = 0.$$

(a) This definition does not require that the random variables are defined on the same probability space. Explain why.

(b) Give an example of $X_n \xrightarrow{d} X$ with $\lim_n \mathbb{P}(X_n \leq x) \neq \mathbb{P}(X \leq x)$ for some x.

(c) What is the difference between the strong law of large numbers and the weak law of large numbers?

10. (5 points) Let random variables $\{X_n\}$ be iid indicators with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 0) = 1/2$.

(a) Show that $S_n = X_1 \cdots X_n$ is a martingale.

(b) Show that $X_n \to 0$ in probability but not in mean.

(c) Does S_n converge a.s.?

11. (5 points) The law of the iterated logarithm. Let $X_1, X_2, \ldots$ be iid random variables with mean 0 and variance 1. Then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_1 + \ldots + X_n}{\sqrt{2n \log\log n}} = 1\right) = 1. \tag{1}$$

(a) Explain the meaning of this statement in terms of typical trajectories of a simple random walk.

(b) Derive from (1) the next result

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{X_1 + \ldots + X_n}{\sqrt{2n \log\log n}} = -1\right) = 1.$$

12. (5 points) The weak law of large numbers.

(a) State the law of large numbers for iid random variables with a finite variance.

(b) Prove the statement. (Hint: either use the Chebyshev inequality or prove first a convergence in mean square.)

13. (5 points) For a fixed integer $n \geq 1$ define a continuous time stochastic process by

$$X_t = \sum_{j=1}^{n}(A_j \cos(jt) + B_j \sin(jt)),$$

where $A_1, B_1, \ldots, A_n, B_n$ are uncorrelated r.v. with zero means and unit variances.

(a) Show that the process X_t is weakly stationary and compute its mean, autocovariance, and autocorrelation functions.

(b) Find the spectral representation for the process X_t.

(c) Under the extra assumption that each of $A_1, B_1, \ldots, A_n, B_n$ has a standard normal distribution, prove that X_t is a strongly stationary process.

14. (5 points) A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$ has the joint characteristic function of the form $\phi(\mathbf{t}) = e^{i\mathbf{t}\boldsymbol{\mu}^{\mathrm{t}} - \frac{1}{2}\mathbf{t}\mathbf{V}\mathbf{t}^{\mathrm{t}}}$. Using this fact

(a) demonstrate that any linear combination $Y = a_1X_1 + \ldots + a_nX_n$ is normally distributed,

(b) what are the mean and the variance of Y?

15. (5 points) There is a theorem saying that given $X_n \xrightarrow{P} X$, convergence $X_n \xrightarrow{L^1} X$ is equivalent to the uniform integrability of the sequence $\{X_n\}$. The uniform integrability means by definition that

$$\sup_n \mathbb{E}(|X_n| I_{\{|X_n|>a\}}) \to 0, \quad a \to \infty.$$

(a) Give an example of a sequence of r.v. that converge in probability but not in mean.

(b) Verify that the sequence in your example is not uniformly integrable.

Partial answers and solutions are also welcome. Good luck!