EXAMPLE OF WRITTEN EXAMINATION II

1 In a school a fire alarm is installed. Without service, the alarm system works for a random time, T [years], which is exponentially distributed with expectation 8 years, i.e. $T \in \text{Exp}(8)$. Inspections take place at regular intervals (once a year), in which the fire alarm is thoroughly tested and all critical parts renewed.

(a) What is the probability that the fire alarm does not work after one year (before the inspection!)? (4 p)

(b) According to the local safety regulation the fire alarm should be functioning with a probability exceeding 0.95. Is this requirement fulfilled? (1 p)

(c) To increase the reliability of the alarm system, a second, identical, fire alarm is installed. The two fire alarms work independently. What is the probability that non of the fire alarms work one year after an inspection? (5 p)

2 In a nuclear power plant a new type of component will be installed. One knows that the lifetime, T [days], of the component is exponentially distributed, yet with unknown intensity $\lambda = 1/\mathsf{E}(T)$. Before installation, the lifetimes of three components were tested with the results 235, 197 and 376 days. Use this information together with an improper prior for λ,

$$f^{pri} = 1/\lambda, \ \lambda > 0,$$

to compute the predicted probability that the installed component will last for 6 months. (10 p)

3 In order to design a new shopping centre, the 100-year value of the amount of rain that may fall during one day is needed. The available historical data was condensed and only the highest measured value, X, of daily rain observed during a year was kept. Suppose that the expectation, $\mathsf{E}[X]$, is estimated to be 100 mm and that the coefficient of variation, $\mathsf{R}[X]$ is 0.2.

(a) Assume first that X is normally distributed. Give an estimate of the 100-year daily rain x_{100}. (3 p)

(b) Next give an estimate of x_{100} assuming that X is Gumbel distributed. (Hint: use $\mathsf{E}[X]$ and $\mathsf{R}[X]$ to estimate parameters a, b in the Gumbel distribution.) (5 p)

(c) Which of these two estimates is most reasonable to use? Give a very short motivation. (2 p)

4 In years 1987-2005 severe storms (significant wave height exceeding 8 meters) has been observed at some location in the North Atlantic. The number of severe storms years 1987 - 2005 are:

0 1 2 2 1 0 3 1 1 3 5 3 1 4 4 3 3 0 5

A person claims that the number of severe storms observed during a year is Poisson distributed. Use a suitable test to check this claim (hypothesis). (20 p)

5 The maximal tension S in a vertical cantilever wooden beam, with circular cross-section, can be obtained using the formula

$$S = \frac{32PL}{\pi D^3}$$

where $L = 10$ m is the length of the beam, D diameter of cross-section and P horizontal force. Here, the force is a random variable with expectation $\mathsf{E}[P] = 1500$ N and variance $\mathsf{V}[P] = 1.1 \cdot 10^6$ N^2. Due to the manufacturing method of the beams, the diameter D is random, with expectation $\mathsf{E}[D] = 0.25$ m and variance $\mathsf{V}[D] = 1 \cdot 10^{-4}$ m^2.

(a) Give the expectation and standard deviation for the maximal tension in the beam, i.e. compute $\mathsf{E}[S]$ and $\mathsf{D}[S]$. (Hint: use Gauss approximation.) (12 p)

(b) The strength of wood depends on its relative water contents and since this depends on the weather, the breakage tension, R_B is considered to be a random variable. Compute the Cornell's safety index for the beam if $\mathsf{E}[R_B] = 46 \cdot 10^6$ Pa and $\mathsf{V}[R_B] = 12 \cdot 10^{12}$ (Pa)^2 . (8 p)

6 In a 12-year period a Swedish insurance company experienced 87 events with wind storm losses. Suppose that storm events form a stationary Poisson stream of events with intensity λ.

(a) Estimate the intensity λ of the Poisson stream of events. (3 p)

(b) Give a 95% confidence interval for λ. (7 p)

The insurance company is especially interested in the intensity (frequency) of storms with damages exceeding 100 million SEK, however not all storms are causing such large damages. The historical data indicates that the damages X, say, are independent and lognormally distributed. The mean and standard deviation of the logarithms of the damages are $m_X^* = 14.5$ and $\sigma_X^* = 1.32$ (milion SEK), i.e. $\ln X$ is estimated to be $\mathrm{N}(2.5, 1.32^2)$.

(c) What is the intensity of storms with damages exceeding 100 million SEK? (5 p)

(d) What is the probability that the company experience at least one storm event with a loss exceeding 100 million SEK in a time period of 10 years? (5 p)

Good luck!

Solution 1):

a) $\mathsf{P}(T<1)=1-\exp(-1/8)=0.118$

b) $1-0.118=0.882<0.95$, No

c) $\mathsf{P}(T_1<1\cap T_2<1)=\mathsf{P}(T_1<1)\mathsf{P}(T_2<1)=(1-\exp(-1/8))^2=0.0138$

Solution 2):

$f_\Lambda^{post}=c\lambda\mathrm{e}^{-235\lambda}\lambda\mathrm{e}^{-197\lambda}\lambda\mathrm{e}^{-376\lambda}\frac{1}{\lambda}\sim$ Gamma(3,808)

$p=\mathsf{P}(T\geq 183)=\exp(-\lambda 183)$, $\mathsf{P}^{\text{pred}}(T\geq 183)=\mathsf{E}[P]=\{\text{Eq. (6.29)}\}=(\frac{808}{808+183})^3=0.542$

Solution 3):

a) X is normal implies that $x_{100}^*=100+\lambda_{0.01}20=147$ mm.

b) If X is Gumbel one has that $V(X)=a^2\pi^2/6$ which gives an estimate of $a^*=15.6$, $\mathsf{E}[X]=b+a\cdot 0.5772$ gives $b^*=91$. Now $x_{100}^*=91-15.6\ln(-\ln(0.99))=163$ mm.

Solution 4):

χ^2-test, 6-classes $N=0$, $N=1$, $N=2$, $N=3$, $N=4$, $N\geq 5$, $\lambda^*=2.211$, $p_i^*=(0.1096,\ 0.2424,\ 0.2679,\ 0.1974,\ 0.1091,\ 0.0736)$, $n_i=(3,5,2,5,2,2)$, $n=19\Rightarrow$ $Q=2.99<\chi^2_{0.05}(6-1-1)=9.49$, Do not reject.

Solution: 5)

a) $\frac{\partial S}{\partial P}=\frac{32L}{\pi D^3}$, $\frac{\partial S}{\partial D}=-\frac{96PL}{\pi D^4}$,

$\mathsf{E}[S]\approx\frac{32\mathsf{E}[P]L}{\pi\mathsf{E}[D]^3}=\frac{32\cdot 1.5\cdot 10^3\cdot 10}{\pi 0.25^3}=9.78\cdot 10^6$ Pa , $\mathsf{V}[S]\approx(\frac{32L}{\pi\mathsf{E}[D]^3})^2\mathsf{V}[P]+(\frac{96\mathsf{E}[P]L}{\pi\mathsf{E}[D]^4})^2\mathsf{V}[D]=4.81\cdot 10^{13}$, $\mathsf{D}[S]=6.94\cdot 10^6$ Pa

b)$\beta_C=\frac{\mathsf{E}[R_B-S]}{\mathsf{D}[R_B-S]}=\frac{\mathsf{E}[R_B]-\mathsf{E}[S]}{\sqrt{\mathsf{V}[R_B]+\mathsf{V}[S]}}=\frac{46\cdot 10^6-9.78\cdot 10^6}{\sqrt{12\cdot 10^{12}+48.1\cdot 10^{12}}}=4.7$

Solution 6): a) $\lambda^*=\frac{x}{n}=\frac{87}{12}=7.25$

b) Two possible solutions

approx.: $\sigma_\varepsilon^*=\sqrt{\lambda^*/n}=0.773$

$\lambda\in[\lambda^*-\lambda_{\alpha/2}\sigma_\varepsilon^*,\lambda^*+\lambda_{\alpha/2}\sigma_\varepsilon^*]=[7.25-1.96\cdot 0.773, 7.25+1.96\cdot 0.773]=[5.73,8.77]$

exact: $\lambda\in[\frac{\chi^2_{1-\alpha/2}(2n\lambda^*)}{2n},\frac{\chi^2_{\alpha/2}(2n\lambda^*+2)}{2n}]=[5.81,8.94]$

c) $\hat{\lambda}^*=\lambda^*\cdot\mathsf{P}(X>100)=0.773\cdot(1-\Phi(\frac{\log(100)-2.5}{1.32}))=0.043$ year^{-1}

d) $N\in\mathrm{Po}(\hat{\lambda}\cdot 10)$, $\mathsf{P}(N\geq 1)=1-\mathsf{P}(N=0)=1-\exp(-\hat{\lambda}^*\cdot 10)=0.35$