Solutions:

1 Gumbel distribution is given by the following cumulative distribution function

$$F(x) = e^{-e^{-(x-b)/a}}, \; a > 0.$$

(a) By the definition of x_{100}, it is the solution in x of

$$F(x) = 99/100,$$

where $a = a^*$ and $b = b^*$. By a straightforward algebra

$$\begin{aligned} -e^{-(x-b^*)/a^*} &= \ln 99/100 \\ -(x-b^*)/a^* &\approx \ln 0.01 \\ (x-b^*)/a^* &\approx 4.61 \\ x &\approx 4.61 * a^* + b^* \\ x &\approx 4.78 \end{aligned}$$

Thus an estimated and approximated value the hundred year sea-level at Point Pirie is x_{100} is $x^*_{100} = 4.78$[m]. Alternatively, one can further approximate the ML estimator (see Section 10.3) by using

$$x^*_{100} = b^* + a^* \ln 100 = 3.87 + 0.198 * 4.6051 = 4.78181.$$

We observe that the both approximations yield the same value.

(b) For finding the confidence interval for x_{100}, we use the normal approximation to the estimation error distribution as given Section 10.3.1 of the Namely, the confidence interval is given by

$$\begin{aligned} x^*_{100} \pm \lambda_{0.005} \cdot a^* \sqrt{(1.11 + 0.61 * \ln^2 100 + 0.52 * \ln 100)/65} \\ \approx 4.78 \pm \lambda_{0.005} \cdot 0.1 \end{aligned}$$

or in other words $[4.5, 5.0]$[m].

2 Given λ, the number of failures X per year is following Poisson distribution with parameter λ, which has the mean $\mathsf{E}[X] = \lambda$.

(a) The average cost is $\mathsf{E}[c \cdot X] = c \cdot \mathsf{E}[X] = c \cdot \lambda = 2000$ SEK/year. We may also approach the problem through Bayesian methodology and assume the conjugate exponential prior for λ with the mean 0.5 which corresponds to Gamma(α, β) with $\alpha = 1$ and $\beta = 2$ so that $\mathsf{E}[\Lambda] = 1/\beta = 0.5$. This approach leads to the same answer $\mathsf{E}[c \cdot X] = \mathsf{E}[\mathsf{E}[c \cdot X|\Lambda]] = c \cdot \mathsf{E}[\Lambda] = 2000$ SEK/year.

(b) Our prior distribution for the problem as described above is Gamma(α, β) with $\alpha = 1$ and $\beta = 2$. because it is conjugate to the Poisson distribution, the posterior distribution of Λ is given by Gamma$(\alpha + x, \beta + t)$ =Gamma$(2, 2.5)$ hence using the mean value of Gamma distribution gives $\mathsf{E}[c\Lambda] = 4000\frac{2}{2.5} = 16000/5 = 3200$ SEK.

3 The standarized death-rates is another name for the failure intensity function. Let T denotes a life-time of a women. We are asked to compute the risk

$$\begin{aligned} \mathsf{P}(T < 90|T > 80) &= 1 - \mathsf{P}(T > 90)/\mathsf{P}(Y > 80) \\ &= 1 - R(90)/R(80) = 1 - \exp(-\int_{80}^{90} \lambda(t)\ dt). \end{aligned}$$

Given the estimated parameters, we compute the integral

$$\begin{aligned} \int_{80}^{90} \alpha^* + \beta^* e^{(t-3)/c^*}\ dt &= 10 * \alpha^* + \beta^* c^* e^{(87-77)/c^*} \\ &= 9 \cdot 10^{-3} + 4.4 \cdot 10.34 \cdot 10^{-5}\left(e^{87/10.34} - e^{77/10.34}\right) \\ &= 1.280636. \end{aligned}$$

Thus the estimated risk is approximately $1 - e^{-1.28} \approx 72\%$.

4 For the problem, we use Barlow-Proschan test that is specifically designed for testing Poisson model (using χ^2-test is also possible but not recommended for this problem). The test statistics (see Section 7.4 of the textbook) for the problem is

$$z = \sum_{k=1}^{n-1} \sum_{i=1}^{k} t_i / \sum_{k=1}^{n} t_k = \sum_{k=1}^{n-1} s_k/s_n,$$

where t_i are times between failures and s_k are failures time. Straightforward algebra gives s_k's as

50	94	196	268	290	329	332	347	544	732	811	899
945	950	955	991	1013	1152	1362	1459	1489	1512	1525	1539

Thus $z = 18245/1539 = 11.8551$. According to the test we would reject the model at the significance level $\alpha = 0.05$ if the value of z falls outside the interval

$$[23/2 - 1.96\sqrt{23/12}, 23/2 + 1.96\sqrt{23/12}] \approx [8.79, 14.21],$$

which is not the case, so we do not have evidence that using the Poisson model for these data is not appropriate.

The estimate of the intensity is then given as the reciprocal of the average of times between failures, i.e. $\lambda^* = 24/1539 \approx 0.0156$ per hour.

5 The problem combines two random factors: occurences of fires in the industrial buildings and losses due to these fires. In Part-a) we consider only the first factor, in Part b) we account only for the second factor, and finaly in Part c), we account for both.

(a) We assume that the we compute annual rate of fires and the obvious estimator (which is also the maximum likelihood estimator) of it is given as

$$\lambda^* = 57/(10 * 285) = 0.02.$$

(b) We use normal approximation to the distributon of the estimation error $\mathcal{E}$ that is discussed in Subsection 4.4.2 of the textbook. The

$$[5.7 - 1.96 * \sqrt{5.7/10}, 5.7 + 1.96 * \sqrt{5.7/10}] = [4.22, 7.17].$$

(c) Here we use the definition of lognormal random variable which states that $Y_i = \log X_i$ is normally distributed, so

$$\begin{aligned} \mathsf{P}(X_i > 10^7) &= \mathsf{P}(Y_i > 7 \log 10) \\ &\approx \mathsf{P}(Y_i > 16.12) = \mathsf{P}((Y_i - m_X)/\sigma_X > (16.12 - m_X)/\sigma_X) \\ &\approx \mathsf{P}(Z > 0.847) = 0.2, \end{aligned}$$

where in the last approximation we have used estimated values of m_X and σ_X, while Z is the standard normal variable.

(d) The risk is

$$1 - \exp(-t\lambda \cdot \mathsf{P}(X > 10^7)) \approx t\lambda \cdot \mathsf{P}(X > 10^7) = 0.02 * 0.2 * 10 = 0.04,$$

i.e. 4%.

6 The failure function is $h(H,S) = 27 - S - H$ where S is the subsidence for a given year.

(a) The failure function is $h(H,S) = 27 - S - H$, $S = 8.5$ hence $P(h(H,S) < 0) = P(H > 18.5) = 1 - e^{-e^{-(18.5-5.2)/1.95}} \approx 1/1000$.

(b) Cornells safety index $I = \mathsf{E}[h(H,S)]/\sqrt{\mathsf{V}(h(H,S))}$ and here $I = (18.5 - \mathsf{E}[H])/\sqrt{\mathsf{V}(h(H,S))} = 4.87$.

(c) Let x be the height one need to lift the deck. The failure function is now $h(H,S) = x + 27 - 13.3 - 0.5\sqrt{X} - H$. We use Gauss' approximation to compute the mean and variance of $h(H,S)$ and we obtain

$$\mathsf{E}[h(H,S)] \approx x + 13.7 - 5.2 - 1.95 * 0.5773 - 0.5\sqrt{1.5} = x + 6.76,$$

while the variance

$$\begin{aligned}\mathsf{V}(h(H,S) &= \mathsf{V}(H) + \mathsf{V}(S)\\ &\approx 1.95^2 * \pi^2/6 + 0.5^2 * (1/(4 * 1.5)) * 0.5 = 6.2757.\end{aligned}$$

The safety index $I \approx (x + 6.76)/\sqrt{6.276}$ should be equal to 4.87. Giving $x = -6.76 + 2.5 * 4.87 = 5.42$ meters.