Probability, Statistics and Risk, MVE300
TABLE of FORMULÆ

Probability theory

Basic probability theory

Let $\mathcal{S}$ be a sample space, and let P be a probability on $\mathcal{S}$. Then, for all events $A, B, A_1, A_2, \dots, A_n \subseteq \mathcal{S}$,

(1) Kolmogorov's axioms

(1.1) $0 \le \mathsf{P}(A) \le 1$
(1.2) $\mathsf{P}(\mathcal{S}) = 1$
(1.3) $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$, if A and B are disjoint.

(2) $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$.

(3) A and B are independent $\iff \mathsf{P}(A \cap B) = \mathsf{P}(A)\,\mathsf{P}(B)$.

(4) Conditional probability: $\mathsf{P}(B \mid A) = \dfrac{\mathsf{P}(A \cap B)}{\mathsf{P}(A)}$.

(5) Law of total probability: $\mathsf{P}(B) = \sum_{i=1}^{n} \mathsf{P}(B \mid A_i)\,\mathsf{P}(A_i)$,

whenever $A_1, \dots, A_n$ are pairwise disjoint and satisfy $\bigcup_{i=1}^{n} A_i = \mathcal{S}$.

(6) Bayes' theorem: $\mathsf{P}(A_i | B) = \dfrac{\mathsf{P}(B|A_i)\mathsf{P}(A_i)}{\mathsf{P}(B)} = \dfrac{\mathsf{P}(B|A_i)\mathsf{P}(A_i)}{\sum_{j=1}^{n} \mathsf{P}(B \mid A_j)\,\mathsf{P}(A_j)}$,

whenever $A_1, \dots, A_n$ are pairwise disjoint and satisfy $\bigcup_{k=1}^{n} A_k = \mathcal{S}$.

One-dimensional random variables

(7) Distribution function for the random variable X: $F_X(x) = \mathsf{P}(X \le x)$.

(8) Probability-mass function for the discrete random variable X: $p_X(x) = \mathsf{P}(X = x)$.

(9) Density function for the continuous random variable X: $f_X(x) = \dfrac{\mathrm{d}F_X(x)}{\mathrm{d}x}$ for all x where F_X is differentiable.

(10) If X is discrete, then

$$\mathsf{P}(a < X \le b) = F_X(b) - F_X(a) = \sum_{\substack{x \in]a;b] \\ p_X(x) \neq 0}} p_X(x)$$

If there is no element x in $]a;b]$ such that $p_X(x) \neq 0$, then $F_X(b) - F_X(a) = 0$.

(11) If X is continuous, then

$$\mathsf{P}(a < X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,\mathrm{d}x$$

Two-dimensional random variables

(12) Joint distribution function for the two random variables X and Y:

$$F_{X,Y}(x,y) = \mathsf{P}(X \le x \cap Y \le y)$$

(13) Joint probability-mass function for the two discrete random variables X and Y:

$$p_{X,Y}(x,y) = \mathsf{P}(X = x \cap Y = y)$$

(14) Joint density function for the two continuous random variables X and Y:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

for all (x,y) where the derivative exists.

(15) If X and Y both are discrete:

$$\mathsf{P}\big((X,Y) \in A\big) = \sum_{\substack{(x,y)\in A \\ p_{X,Y}(x,y)\neq 0}} p_{X,Y}(x,y)$$

If there is no pair (x,y) in A such that $p_X(x) \neq 0$, then $\mathsf{P}\big((X,Y) \in A\big) = 0$.

(16) If X and Y both are continuous:

$$\mathsf{P}\big((X,Y) \in A\big) = \int_{(x,y)\in A} f_{X,Y}(x,y)\,\mathrm{d}(x,y)$$

Conditional distributions

(17) Conditional distribution function: $F_{X|Y}(x,y) = \mathsf{P}(X \le x \,|\, Y = y)$.

(18) Conditional probability-mass function for the discrete random variable X:

$$p_{X|Y}(x,y) = \mathsf{P}(X = x \,|\, Y = y)$$

If Y is also discrete, then $p_{X|Y}(x,y) = \begin{cases} \frac{p_{X,Y}(x,y)}{p_Y(y)}, & p_Y(y) \neq 0 \\ 0, & p_Y(y) = 0. \end{cases}$

(19) Conditional density function for the continuous random variable X:

$$f_{X|Y}(x,y) = \frac{\partial F_{X|Y}(x,y)}{\partial x}$$

If Y is also continuous, then $f_{X|Y}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & f_Y(y) \neq 0 \\ 0, & f_Y(y) = 0. \end{cases}$

(20) Bayes' theorem:

(20.1) X discrete and Y discrete: $p_{Y|X}(y,x) = \begin{cases} \frac{p_{X|Y}(x,y)\cdot p_Y(y)}{p_X(x)}, & p_X(x) \neq 0, \\ 0, & p_X(x) = 0. \end{cases}$

(20.2) X discrete and Y continuous: $f_{Y|X}(y,x) = \begin{cases} \frac{p_{X|Y}(x,y)\cdot f_Y(y)}{p_X(x)}, & p_X(x) \neq 0, \\ 0, & p_X(x) = 0. \end{cases}$

(20.3) X continuous and Y discrete: $p_{Y|X}(y,x) = \begin{cases} \frac{f_{X|Y}(x,y)\cdot p_Y(y)}{f_X(x)}, & f_X(x) \neq 0, \\ 0, & f_X(x) = 0. \end{cases}$

(20.4) X continuous and Y continuous: $f_{Y|X}(y,x) = \begin{cases} \frac{f_{X|Y}(x,y)\cdot f_Y(y)}{f_X(x)}, & f_X(x) \neq 0, \\ 0, & f_X(x) = 0. \end{cases}$

(21) Marginal probability-mass function for the discrete random variable X:

(21.1) Y is discrete: $p_X(x) = \sum_{\substack{y \\ p_Y(y)\neq 0}} p_{X|Y}(x,y)\cdot p_Y(y) = \sum_{\substack{y \\ p_{X,Y}(x,y)\neq 0}} p_{X,Y}(x,y)$

(21.2) Y is continuous: $p_X(x) = \int_{-\infty}^{\infty} p_{X|Y}(x,y)\cdot f_Y(y)\,\mathrm{d}y$

(22) Marginal density function for the continuous random variable X:

(22.1) Y is discrete: $f_X(x) = \sum_{\substack{y \\ p_Y(y)\neq 0}} f_{X|Y}(x,y)\cdot p_Y(y)$

(22.2) Y is continuous: $f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x,y)\cdot f_Y(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$

(23) If X and Y are independent,
(23.1) then $F_{X,Y}(x,y) = F_X(x)\cdot F_Y(y)$,
(23.2) then $p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)$ if X and Y are discrete,
(23.3) then $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$ if X and Y are continuous,
(23.4) then $F_{X|Y}(x,y) = F_X(x)$,
(23.5) then $p_{X|Y}(x,y) = p_X(x)$ if X is discrete,
(23.6) then $f_{X|Y}(x,y) = f_X(x)$ if X is continuous.

Law of total probability again

Law of total probability: Let A be an event.

(24) If X is a discrete random variable, then $\mathsf{P}(A) = \sum_{p_X(x)\neq 0} \mathsf{P}(A|X=x)\cdot p_X(x)$.

(25) If X is a continuous random variable, then $\mathsf{P}(A) = \int_{-\infty}^{\infty} \mathsf{P}(A|X=x)\cdot f_X(x)\,\mathrm{d}x$.

Expectation, variance, and the like

(26) Let g be a real-valued function $x \curvearrowright g(x)$. Then the expectation of $g(X)$ is given by

(26.1) $\mathsf{E}\big(g(X)\big) = \sum_{p_X(x)\neq 0} g(x)\,p_X(x)$, if X is discrete,

(26.2) $\mathsf{E}\big(g(X)\big) = \int_{-\infty}^{\infty} g(x)\,f_X(x)\,\mathrm{d}x$, if X is continuous.

(27) Let g be a real-valued function $(x,y) \curvearrowright g(x,y)$. Then the expectation of $g(X,Y)$ is given by

(27.1) $\mathsf{E}\big(g(X,Y)\big) = \sum_{p_{X,Y}(x,y)\neq 0} g(x,y)\,p_{X,Y}(x,y)$, if X and Y are discrete, (27.2) $\mathsf{E}\big(g(X,Y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,\mathrm{d}(x,y)$, if X and Y are continuous.

(28) Variance: $\mathsf{V}(X) = \mathsf{E}\Big(\big(X-\mathsf{E}(X)\big)^2\Big) = \mathsf{E}(X^2) - \big(\mathsf{E}(X)\big)^2$.

(29) Standard deviation: $\mathsf{D}(X) = \sqrt{\mathsf{V}(X)}$.

(30) Coefficient of variation: $\mathsf{R}(X) = \mathsf{D}(X)/\mathsf{E}(X)$.

(31) Covariance: $\mathsf{C}(X;Y) = \mathsf{E}\Big(\big(X-\mathsf{E}(X)\big)\big(Y-\mathsf{E}(Y)\big)\Big) = \mathsf{E}(XY) - \mathsf{E}(X)\,\mathsf{E}(Y)$.

(32) $\mathsf{C}(X;X) = \mathsf{V}(X)$.

(33) Coefficient of correlation: $\rho(X,Y) = \dfrac{\mathsf{C}(X,Y)}{\mathsf{D}(X)\,\mathsf{D}(Y)}$.

(34) Expectation is linear, i.e. $\mathsf{E}(aX+bY) = a\mathsf{E}(X) + b\mathsf{E}(Y)$.

(35) $\mathsf{V}(aX \pm bY) = a^2\mathsf{V}(X) + b^2\mathsf{V}(Y) \pm 2ab\,\mathsf{C}(X,Y)$.

(36) Covariance is bilinear, i.e. $\mathsf{C}(aX+bY, cZ) = ac\mathsf{C}(X,Z) + bc\mathsf{C}(Y,Z)$ and $\mathsf{C}(cZ, aX+bY) = ca\mathsf{C}(Z,X) + cb\mathsf{C}(Z,Y)$.

(37) For independent random variables X, Y: $\mathsf{E}(XY) = \mathsf{E}(X)\,\mathsf{E}(Y)$.

(38) Gauss' approximations: Let g be a real-valued function $(x_1, x_2, \ldots, x_n) \curvearrowright g(x_1, x_2, \ldots, x_n)$. Then

$$\begin{aligned} \mathsf{E}(g(X_1,\ldots,X_n)) &\approx g(\mathsf{E}(X_1),\ldots,\mathsf{E}(X_n)). \\ \mathsf{V}(g(X_1,\ldots,X_n)) &\approx \sum_{i=1}^{n} c_i^2\mathsf{V}(X_i) + 2\sum_{\substack{1\le i\le n\\ 1\le j\le n\\ i<j}} c_i c_j \mathsf{C}(X_i,\, X_j), \end{aligned}$$

where $c_i = g'_{x_i}(\mathsf{E}(X_1),\ldots,\mathsf{E}(X_n)) = \dfrac{\partial g}{\partial x_i}(\mathsf{E}(X_1),\ldots,\mathsf{E}(X_n))$.

Normal (Gaussian) distribution

(39) Univariate normal (Gaussian) distribution ($\sigma > 0$):

$$X \in \mathrm{N}(m,\sigma^2) \quad \Leftrightarrow \quad \frac{X-m}{\sigma} \in \mathrm{N}(0,1)$$

(40) Bivariate normal (Gaussian) distribution: Let m_1, m_2, σ_1, σ_2, and ϱ be real numbers ($\sigma_1 > 0$, $\sigma_2 > 0$, $-1 < \varrho < 1$). If $(X,Y) \in \mathrm{N}(m_1, m_2, {\sigma_1}^2, {\sigma_2}^2, \varrho)$, then

(40.1) $f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}}\,\mathrm{e}^{-\frac{1}{2}\cdot\left(\frac{(x-m_1)^2}{{\sigma_1}^2} + \frac{(y-m_2)^2}{{\sigma_2}^2} - 2\varrho\cdot\frac{x-m_1}{\sigma_1}\cdot\frac{y-m_2}{\sigma_2}\right)}$,

(40.2) $X \in \mathrm{N}(m_1, {\sigma_1}^2)$, $Y \in \mathrm{N}(m_2, {\sigma_2}^2)$, $\mathsf{C}(X,Y) = \varrho\sigma_1\sigma_2$, $\rho(X,Y) = \varrho$,

(40.3) $f_{X|Y}(x,y) = \dfrac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\varrho^2}}\,\mathrm{e}^{-\frac{1}{2}\cdot\frac{1}{{\sigma_1}^2(1-\varrho^2)}\cdot\left(x-\left(m_1+\varrho\frac{\sigma_1}{\sigma_2}(y-m_2)\right)\right)^2}$,
i.e. a $\mathrm{N}\big(m_1 + \varrho\frac{\sigma_1}{\sigma_2}(y-m_2), {\sigma_1}^2(1-\varrho^2)\big)$ distribution,

(40.4) $aX + bY \in \mathrm{N}(am_1 + bm_2, a^2{\sigma_1}^2 + b^2{\sigma_2}^2 + 2ab\varrho\sigma_1\sigma_2)$ for all real numbers a and b.

Limit theorems

(41) Law of Large Numbers (LLN):
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with existing expectation $\mathsf{E}(X_i) = m$. Then

$$Y_n = \frac{X_1 + \ldots + X_n}{n} \to \mathsf{E}(X_i),$$

då $n \to \infty$.

(42) Central Limit Theorem:
Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with existing expectation $\mathsf{E}(X_i)$ and existing standard deviation $\mathsf{D}(X_i) = \sigma < \infty$. Then

$$Y_n = X_1 + \ldots + X_n \in \mathrm{AsN}(n{\cdot}m,\ n{\cdot}\sigma^2),$$

when $n \to \infty$.

(43) We have approximately

(43.1)	$\mathrm{Bin}(n,p)$	$\approx$	$\mathrm{Po}(np)$	if	$p \le \frac{1}{10}$ and $n \ge 10$.
(43.2)	$\mathrm{Bin}(n,p)$	$\approx$	$\mathrm{N}(np,\ np(1-p))$	if	$np(1-p) \ge 10$.
(43.3)	$\mathrm{Po}(m)$	$\approx$	$\mathrm{N}(m,\ m)$	if	$m \ge 15$.

Sums of random variables

(44) Let $X_1 \in \mathrm{N}(m_1, {\sigma_1}^2),\ \ldots,\ X_n \in \mathrm{N}(m_n, {\sigma_n}^2)$ be n independent, normally distributed random variables. For any set $c_1, \ldots, c_n$ of n real numbers, we have

$$\sum_{i=1}^{n} c_i X_i \in \mathrm{N}\left(\sum_{i=1}^{n} c_i m_i, \sum_{i=1}^{n} {c_i}^2 {\sigma_i}^2\right).$$

(45) If X_1 and X_2 are independent, then

(45.1)	$X_1 \in \mathrm{Bin}(n_1,p),\ X_2 \in \mathrm{Bin}(n_2,p)$	$\Rightarrow$	$X_1 + X_2 \in \mathrm{Bin}(n_1+n_2,p)$.
(45.2)	$X_1 \in \mathrm{Po}(m_1),\ X_2 \in \mathrm{Po}(m_2)$	$\Rightarrow$	$X_1 + X_2 \in \mathrm{Po}(m_1+m_2)$.
(45.3)	$X_1 \in \mathrm{Gamma}(a_1,b),\ X_2 \in \mathrm{Gamma}(a_2,b)$	$\Rightarrow$	$X_1 + X_2 \in \mathrm{Gamma}(a_1+a_2,b)$.
(45.4)	$X_1 \in \chi^2(f_1),\ X_2 \in \chi^2(f_2)$	$\Rightarrow$	$X_1 + X_2 \in \chi^2(f_1+f_2)$.

Statistics

Point estimation

Let $x_1, \ldots, x_n$ be observations of n independent, identically distributed random variables with expectation m and standard deviation σ. Then unbiased estimations of m and σ^2 are given by

(46) $m^* = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i = \bar{x}$

(47) $(\sigma^2)^* = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i - m)^2, \quad m$ known.

(48) $(\sigma^2)^* = s^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad m$ unknown.

Confidence intervals

(49) Let ϑ be some parameter, and let Θ^* (r.v.) be an estimator of ϑ such that Θ^* is (approximately) normally distributed with expectation ϑ. Let ϑ^* be the estimate of ϑ, i.e. let ϑ^* be the observation of Θ^*. Then

$$I_\vartheta = [\vartheta^* - \lambda_{(1-\gamma)/2}\cdot \mathsf{d}(\Theta^*); \vartheta^* + \lambda_{(1-\gamma)/2}\cdot \mathsf{d}(\Theta^*)] \quad \text{(two-sided)},$$
$$I_\vartheta = [\vartheta^* - \lambda_{1-\gamma}\cdot \mathsf{d}(\Theta^*); \infty] \quad \text{(one-sided, bounded below)},$$
$$I_\vartheta = [-\infty; \vartheta^* + \lambda_{1-\gamma}\cdot \mathsf{d}(\Theta^*)] \quad \text{(one-sided, bounded above)}$$

are confidence intervals for ϑ with approximative confidence level γ (γ is typically "large", $\gamma = 0{,}95$, $\gamma = 0{,}99$, ...). Here, $\mathsf{d}(\Theta^*)$ is the standard error of the estimator Θ^*.

$$\mathsf{d}(\Theta^*) = \begin{cases} = \mathsf{D}(\Theta^*) & \text{if } \mathsf{D}(\Theta^*) \text{ is known and independent of } \vartheta, \\ = \big(\mathsf{D}(\Theta^*)\big)^* & \text{if } \mathsf{D}(\Theta^*) \text{ is unknown or dependent on } \vartheta. \end{cases}$$

Examples (ϑ is the parameter):

(49.1) Observations: $x_1, \ldots, x_n$.
Random variables lying behind the observations: $X_1, \ldots, X_n$.
Model: $X_1, \ldots, X_n$ are IID with $\mathsf{E}(X_1) = \ldots = \mathsf{E}(X_n) = \vartheta$ and $\mathsf{D}(X_1) = \ldots = \mathsf{D}(X_n) = \sigma$. σ is known and independent of ϑ. The number n is "large".
Estimate, standard error: $\boxed{\vartheta^* = \bar{x},\ \mathsf{d}(\Theta^*) = \dfrac{\sigma}{\sqrt{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$.

(49.2) Observations: $x_1, \ldots, x_n$.
Random variables lying behind the observations: $X_1, \ldots, X_n$.
Model: $X_1, \ldots, X_n$ are IID with $\mathsf{E}(X_1) = \ldots = \mathsf{E}(X_n) = \vartheta$ and $\mathsf{D}(X_1) = \ldots = \mathsf{D}(X_n) = \sigma$. σ is unknown but independent of ϑ. The number n is "large".
Estimate, standard error: $\boxed{\vartheta^* = \bar{x},\ \mathsf{d}(\Theta^*) = \dfrac{s}{\sqrt{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$, $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2}$.

(49.3) Observations: $x_1, \ldots, x_n$.
Random variables lying behind the observations: $X_1, \ldots, X_n$.
Model: $X_1 \in \mathrm{N}(\vartheta, \sigma^2), \ldots, X_n \in \mathrm{N}(\vartheta, \sigma^2)$ are IID. The standard deviation σ is known and independent of ϑ. Of course, $\sigma > 0$.
Estimate, standard error: $\boxed{\vartheta^* = \bar{x},\ \mathsf{d}(\Theta^*) = \dfrac{\sigma}{\sqrt{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$.
Comment: The intervals will be exact.

(49.4) Observations: $x_1, \ldots, x_n$.
Random variables lying behind the observations: $X_1, \ldots, X_n$.
Model: $X_1 \in \mathrm{N}(\vartheta, \sigma^2), \ldots, X_n \in \mathrm{N}(\vartheta, \sigma^2)$ are IID. The standard deviation σ is unknown but independent of ϑ. Of course, $\sigma > 0$.
Estimate, standard error: $\boxed{\vartheta^* = \bar{x},\ \mathsf{d}(\Theta^*) = \dfrac{s}{\sqrt{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$, $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2}$.
Comment: The intervals will be exact if the standard-normal quantile is replaced by a $\mathrm{t}(n-1)$ quantile. Otherwise, the approximative interval is inaccurate unless n is "large".

(49.5) Observations: $x_1, \ldots, x_n$.
Random variables lying behind the observations: $X_1, \ldots, X_n$.
Model: $X_1 \in \mathrm{Po}(\vartheta), \ldots, X_n \in \mathrm{Po}(\vartheta)$ are IID. It is so that $n \cdot \vartheta \geq 15$.

Estimate, standard error: $\boxed{\vartheta^* = \bar{x},\ \mathsf{d}(\Theta^*) = \sqrt{\frac{\bar{x}}{n}}}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Comment: It must be verified that $n \cdot \vartheta^* \geq 15$; then use that $\sum_{i=1}^{n} x_i = n \cdot \vartheta^*$.

(49.6) Observation: x.
Random variable lying behind the observation: X.
Model: $X \in \mathrm{Bin}(n, \vartheta)$. It is so that $n \cdot \vartheta \cdot (1 - \vartheta) \geq 10$.

Estimate, standard error: $\boxed{\vartheta^* = \frac{x}{n},\ \mathsf{d}(\Theta^*) = \sqrt{\frac{\vartheta^* \cdot (1-\vartheta^*)}{n}}}$.

Comment: It must be verified that $n \cdot \vartheta^* \cdot (1 - \vartheta^*) \geq 10$; then use that $x \cdot (n - x)/n = n \cdot \vartheta^* \cdot (1 - \vartheta^*)$.

Hypothesis testing

(50) Let ϑ be a parameter. We want to test the simple hypothesis

$$H_0 : \qquad \vartheta = \vartheta_0$$

on the significance level α (α is typically "small", $\alpha = 0{,}05$, $\alpha = 0{,}01, \ldots$). Then the test will be

$$\begin{cases} \text{Reject } H_0 & \Leftrightarrow \quad \vartheta_0 \notin I \\ \text{Do not reject } H_0 & \Leftrightarrow \quad \vartheta_0 \in I \end{cases}$$

where I is a confidence interval of ϑ with (approximative) confidence level $1 - \alpha$.

If $H_1 :\ \vartheta \neq \vartheta_0$, then choose I to be two-sided.
If $H_1 :\ \vartheta > \vartheta_0$, then choose I to be one-sided and bounded below.
If $H_1 :\ \vartheta < \vartheta_0$, then choose I to be one-sided and bounded above.

(51) The χ^2 test. Let H_0 be a hypothesis about the distribution, expressed by probabilities $p_1, \ldots, p_r$. We have n observations. Calculate

$$Q = \sum_{i=1}^{r} \frac{(x_i - np_i)^2}{np_i}.$$

Reject H_0 if $Q > \chi^2_\alpha(r - 1)$.

Bayesian updating

Let Θ be the parameter modelled as a random variable. Let X be the observed random variable with observed value x.

(52) Θ is discrete, X is discrete: $p_\Theta^{\text{post}}(\vartheta) = c\,\mathsf{P}(X = x | \Theta = \vartheta) \cdot p_\Theta^{\text{prior}}(\vartheta)$,
where $c^{-1} = \mathsf{P}(X = x) = \sum_{\substack{\vartheta \\ p_\Theta^{\text{prior}}(\vartheta) \neq 0}} \mathsf{P}(X = x | \Theta = \vartheta) \cdot p_\Theta^{\text{prior}}(\vartheta)$.

(53) Θ is discrete, X is continuous: $p_\Theta^{\text{post}}(\vartheta) = c\, f_{X|\Theta}(x; \vartheta) \cdot p_\Theta^{\text{prior}}(\vartheta)$,
where $c^{-1} = f_X(x) = \sum_{\substack{\vartheta \\ p_\Theta^{\text{prior}}(\vartheta) \neq 0}} f_{X|\Theta}(x; \vartheta) \cdot p_\Theta^{\text{prior}}(\vartheta)$.

(54) Θ is continuous, X is discrete: $f_\Theta^{\text{post}}(\vartheta) = c\,\mathsf{P}(X = x|\Theta = \vartheta) \cdot f_\Theta^{\text{prior}}(\vartheta)$, where $c^{-1} = \mathsf{P}(X = x) = \int_{-\infty}^{\infty} \mathsf{P}(X = x|\Theta = \vartheta) \cdot f_\Theta^{\text{prior}}(\vartheta)\,\mathrm{d}\vartheta$.

(55) Θ is continuous, X is continuous: $f_\Theta^{\text{post}}(\vartheta) = c\,f_{X|\Theta}(x;\vartheta) \cdot f_\Theta^{\text{prior}}(\vartheta)$, where $c^{-1} = f_X(x) = \int_{-\infty}^{\infty} f_{X|\Theta}(x;\vartheta) \cdot f_\Theta^{\text{prior}}(\vartheta)\,\mathrm{d}\vartheta$.

(56) Particularly,

	Prior distribution of Θ	Conditional distribution of X, given $\Theta = \vartheta$	Posterior distribution of Θ (x is the observation of X)
(56.1)	Gamma(a, b)	Po$(\vartheta \cdot t)$	Gamma$(a + x, b + t)$
(56.2)	$\beta(a, b)$	Bin(n, ϑ)	$\beta(a + x, b + n - x)$

(57) If the event A and the random variable X are independent on condition that Θ is known, then we have the law of total probability:

(57.1) $\quad \mathsf{P}^{\text{post}}(A) = \sum_{p_\Theta^{\text{post}}(\vartheta) \neq 0} \mathsf{P}(A|\Theta = \vartheta) \cdot p_\Theta^{\text{post}}(\vartheta),\qquad$ if Θ is discrete,

(57.2) $\quad \mathsf{P}^{\text{post}}(A) = \int_{-\infty}^{\infty} \mathsf{P}(A|\Theta = \vartheta) \cdot f_\Theta^{\text{post}}(\vartheta)\,\mathrm{d}\vartheta,\qquad$ if Θ is continuous.

(58) If we have observed the occurrence of an event B (instead of having observed the random variable X), then:

(58.1) $\quad p_\Theta^{\text{post}}(\vartheta) = \mathsf{P}(B \mid \Theta = \vartheta) \cdot p_\Theta^{\text{prior}}(\vartheta),\qquad$ if Θ is discrete,

(58.2) $\quad f_\Theta^{\text{post}}(\vartheta) = \mathsf{P}(B \mid \Theta = \vartheta) \cdot f_\Theta^{\text{prior}}(\vartheta),\qquad$ if Θ is continuous.

If the event A and the event B are independent on condition that Θ is known, then (??.1) and (??.2) still are valid.

Miscellaneous

The Poisson process

Let $N(t)$ be the number of events taking place in the time interval $]0;t]$. If $N(t)$ is a Poisson process with constant intensity λ, then

(59) $N(t) \in \text{Po}(\lambda t)$.

(60) Time lags between consecutive events are independent and exponentially distributed with expectation $1/\lambda$.

(61) The number of events occurring in a time interval I_1 and the number of events occurring in another time interval I_2 are independent if I_1 and I_2 are disjoint.

Failure rate

Let T be a positive, continuous random variable with density function f_T and distribution function F_T.

(62) $\lambda(t) = \dfrac{f_T(t)}{1 - F_T(t)}, \qquad t \geq 0$ and $F_T(t) \neq 1$.

(63) $\mathsf{P}(T > t) = \exp\left(-\int_0^t \lambda(s)\,\mathrm{d}s\right), \qquad t \geq 0$

(64) $\mathsf{P}(t < T \leq t + \Delta \mid T > t) \approx \lambda(t) \cdot \Delta$, if Δ is "small" ($\Delta > 0$).

Quantiles

Quantiles is the same as fractiles. Let α be a real number such that $0 < \alpha < 1$. Let X be a continuous random variable with distribution function F_X.

(65) The α-quantile (denoted x_α) is defined to be any number such that

$$\mathsf{P}(X > x_\alpha) = \alpha \qquad \text{or, equivalently,} \qquad F_X(x_\alpha) = 1 - \alpha.$$

(66) $x_{1/4} = x_{0,25}$ is called the upper (distribution) quartile.
$x_{1/2} = x_{0,5}$ is called the (distribution) median.
$x_{3/4} = x_{0,75}$ is called the lower (distribution) quartile.
$x_{0,01}, x_{0,02}, \ldots, x_{0,98}, x_{0,99}$ are called the (distribution) percentiles.

(67) Quantiles λ_α based on the standard-normal distribution are denoted λ_α: If $X \in \mathrm{N}(0;1)$, then

$$\mathsf{P}(X > \lambda_\alpha) = \alpha \qquad \Leftrightarrow \qquad \Phi(\lambda_\alpha) = 1 - \alpha \qquad \Leftrightarrow \qquad \lambda_\alpha = \Phi^{-1}(1 - \alpha),$$

where $\Phi^{-1}(\ldots)$ is the inverse function of Φ. (So, $\Phi^{-1}(\ldots)$ has nothing to do with $\dfrac{1}{\Phi(\ldots)}$.)

(67.1) Examples:
$\alpha = 0{,}1 \quad \Rightarrow \quad \lambda_\alpha = 1{,}281\,551\ldots$
$\alpha = 0{,}05 \quad \Rightarrow \quad \lambda_\alpha = 1{,}644\,853\ldots$
$\alpha = 0{,}025 \quad \Rightarrow \quad \lambda_\alpha = 1{,}959\,963\ldots$
$\alpha = 0{,}01 \quad \Rightarrow \quad \lambda_\alpha = 2{,}326\,347\ldots$

(67.2) $\lambda_{1-\alpha} = -\lambda_\alpha \qquad$ (for all α such that $0 < \alpha < 1$)

Cornell's reliability index

Let $h(R_1;\ldots;R_k;S_1;\ldots;S_n)$ be the failure function of k random strength variables $R_1, \ldots, R_k$ and n random load variables $S_1, \ldots, S_n$. Let $\mathsf{E}\big(h(R_1;\ldots;R_k;S_1;\ldots;S_n)\big) > 0$.

(68) Cornell's safety index β_{C} is defined as

$$\beta_{\mathrm{C}} = \frac{\mathsf{E}\big(h(R_1;\ldots;R_k;S_1;\ldots;S_n)\big)}{\mathsf{D}\big(h(R_1;\ldots;R_k;S_1;\ldots;S_n)\big)}.$$

(69) If $h(R_1;\ldots;R_k;S_1;\ldots;S_n)$ is normally distributed, then the probability P_{f} of failure is

$$P_{\mathrm{f}} = \mathsf{P}\big(h(R_1;\ldots;R_k;S_1;\ldots;S_n) \leq 0\big) = 1 - \Phi(\beta_{\mathrm{C}}).$$

(70) An upper bound for the probability P_{f} of failure is

$$P_{\mathrm{f}} = \mathsf{P}\big(h(R_1;\ldots;R_k;S_1;\ldots;S_n) \leq 0\big) \leq \frac{1}{1 + {\beta_{\mathrm{C}}}^2}$$

Log-normal distribution

(71) Let X be a log-normally distributed random variable, i.e. $\ln X \in \mathrm{N}(m, \sigma^2)$ or $\ln(\frac{X}{x_{1/2}}) \in \mathrm{N}(0, \sigma^2)$. Then the coefficient of variation is given by

$$\frac{\mathsf{D}(X)}{\mathsf{E}(X)} = \sqrt{\mathrm{e}^{\sigma^2} - 1}$$

(72) Let X_1 and X_2 be two independent random variables. Then

$$\ln X_1 \in \mathrm{N}(m_1, {\sigma_1}^2),\ \ln X_2 \in \mathrm{N}(m_2, {\sigma_2}^2) \qquad \Rightarrow \qquad \ln({X_1}^{k_1} \cdot {X_2}^{k_2}) \in \mathrm{N}(k_1 m_1 + k_2 m_2, {k_1}^2 {\sigma_1}^2 + {k_2}^2 {\sigma_2}^2)$$

Maximum and minimum

Let $X_1, \ldots, X_n$ be n independent, identically distributed random variables with distribution function $F_X(x)$. If we define

$$X_{\max} = \max(X_1, \ldots, X_n) \quad \text{and} \quad X_{\min} = \min(X_1, \ldots, X_n),$$

then

(73) $F_{X_{\max}}(z) = \big(F_X(z)\big)^n,$

(74) $F_{X_{\min}}(z) = 1 - \big(1 - F_X(z)\big)^n.$

Table of distributions

Distribution		Parameter restrictions	Expectation	Variance
(75) Hypergeometric distribution	$p(x)=\begin{cases}\frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}; & x\in \mathsf{Z}_{[n-\min(n;N_2);\min(n;N_1)]}\\ 0; & \text{otherwise}\end{cases}$	$N_1\in\mathsf{Z}_{\geq 1}$ $N_2\in\mathsf{Z}_{\geq 1}$ $n\in\mathsf{Z}_{[1;N_1+N_2]}$	$n\cdot\frac{N_1}{N_1+N_2}$	$\frac{N_1+N_2-n}{N_1+N_2-1}\cdot n\cdot\frac{N_1\,N_2}{(N_1+N_2)^2}$
(76) Binomial distribution, $\mathrm{Bin}(n,p)$	$p(x)=\begin{cases}\binom{n}{x}p^x(1-p)^{n-x}; & x\in\{0,1,\ldots,n\}\\ 0; & \text{otherwise}\end{cases}$	$n\in\mathsf{Z}_{\geq 1}$ $0<p<1$	np	$np(1-p)$
(77) Poisson distribution, $\mathrm{Po}(m)$	$p(x)=\begin{cases}\mathrm{e}^{-m}\frac{m^x}{x!}; & x\in\mathsf{Z}_{\geq 0}\\ 0, & \text{otherwise}\end{cases}$	$m>0$	m	m
(78) Geometric distribution, $\mathrm{Ge}(p)$	$p(x)=\begin{cases}p(1-p)^x; & x\in\mathsf{Z}_{\geq 0}\\ 0; & \text{otherwise}\end{cases}$	$0<p<1$	$\dfrac{1-p}{p}$	$\dfrac{1-p}{p^2}$
(79) First success distribution	$p(x)=\begin{cases}p(1-p)^{x-1}; & x\in\mathsf{Z}_{\geq 1}\\ 0; & \text{otherwise}\end{cases}$	$0<p<1$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
(80) Uniform distribution, $\mathrm{U}(a,b)$	$f(x)=\begin{cases}\frac{1}{b-a}; & a<x<b\\ 0; & \text{otherwise}\end{cases}$ $F(x)=\begin{cases}0; & x\leq a\\ \frac{x-a}{b-a}; & a<x<b\\ 1; & x\geq b\end{cases}$	$a<b$	$\dfrac{a+b}{2}$	$\dfrac{(a-b)^2}{12}$
(81) Beta distribution[1], $\beta(a,b)$	$f(x)=\begin{cases}\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\,x^{a-1}\,(1-x)^{b-1}; & 0<x<1\\ 0; & \text{otherwise}\end{cases}$ $F(x)=\begin{cases}0; & x\leq 0\\ \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\,\mathrm{B}(x;a;b); & 0<x<1\\ 1; & x\geq 1\end{cases}$	$a>0,\ b>0$	$\dfrac{a}{a+b}$	$\dfrac{ab}{(a+b)^2\,(a+b+1)}$
(82) Normal (Gaussian) distribution[2], $\mathrm{N}(m,\sigma^2)$	$f(x)=\dfrac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(x-m)^2}{2\,\sigma^2}}$ $F(x)=\Phi\left(\frac{x-m}{\sigma}\right)$	$\sigma>0$	m	σ^2
(83) Log-normal distribution[2], $\ln X\in\mathrm{N}(m,\sigma^2)$	$F(x)=\begin{cases}0; & x\leq 0\\ \Phi(\frac{\ln x-m}{\sigma}); & x>0\end{cases}$	$\sigma>0$	$\mathrm{e}^{m+\sigma^2/2}$	$\mathrm{e}^{2m+2\sigma^2}-\mathrm{e}^{2m+\sigma^2}$
(84) Log-normal distribution[2, 3], $\ln\frac{X}{x_{1/2}}\in\mathrm{N}(0,\sigma^2)$	$F(x)=\begin{cases}0; & x\leq 0\\ \Phi\left(\frac{1}{\sigma}\ln\frac{x}{x_{1/2}}\right); & x>0\end{cases}$	$x_{1/2}>0,\ \sigma>0$	$x_{1/2}\cdot\mathrm{e}^{\sigma^2/2}$	${x_{1/2}}^2\cdot(\mathrm{e}^{2\sigma^2}-\mathrm{e}^{\sigma^2})$

[1] Γ is the gamma function; cf. (??). $\mathrm{B}(x;a;b)$ is the incomplete beta function; cf. (??).

[2] $\Phi(x)$ is tabulated in (??).

[3] Here, $x_{1/2}$ denotes the distribution median of the random variable X.

Distribution		Parameter restrictions	Expectation	Variance
(85) Gamma distribution[1], Gamma(a,b)	$f(x)=\begin{cases}0; & x\le 0\\ \frac{b}{\Gamma(a)}\,(b\cdot x)^{a-1}\,\mathrm{e}^{-b\cdot x}; & x\ge 0\end{cases}$ $F(x)=\begin{cases}0; & x\le 0\\ 1-\frac{\Gamma(a;b\cdot x)}{\Gamma(a)}; & x>0\end{cases}$	$a>0,\ b>0$	$\dfrac{a}{b}$	$\dfrac{a}{b^2}$
(86) Exponential distribution, Exp(a)	$F(x)=\begin{cases}0; & x\le 0\\ 1-\mathrm{e}^{-x/a}; & x>0\end{cases}$	$a>0$	a	a^2
(87) Gumbel (type I extreme value) distribution[2]	$F(x)=\mathrm{e}^{-\mathrm{e}^{-(x-b)/a}}$	$a>0$	$b+\gamma a$	$\dfrac{a^2\pi^2}{6}$
(88) Fréchet (type II extreme value) distribution[3, 4]	$F(x)=\begin{cases}0; & x\le b\\ \mathrm{e}^{-(\frac{x-b}{a})^{-c}}; & x>b\end{cases}$	$a>0,\ c>0$	$b+a\Gamma(1-1/c)$	$a^2\left[\Gamma(1-\dfrac{2}{c})-\left(\Gamma(1-\dfrac{1}{c})\right)^2\right]$
(89) Type III extreme value distribution[3, 5]	$F(x)=\begin{cases}\mathrm{e}^{-(-\frac{x-b}{a})^{c}}; & x<b\\ 1; & x\ge b\end{cases}$	$a>0,\ c>0$	$b-a\Gamma(1+1/c)$	$a^2\left[\Gamma(1+\dfrac{2}{c})-\left(\Gamma(1+\dfrac{1}{c})\right)^2\right]$
(90) Weibull distribution[3]	$F(x)=\begin{cases}0; & x\le b\\ 1-\mathrm{e}^{-(\frac{x-b}{a})^{c}}; & x>b\end{cases}$	$a>0,\ c>0$	$b+a\Gamma(1+1/c)$	$a^2\left[\Gamma(1+\dfrac{2}{c})-\left(\Gamma(1+\dfrac{1}{c})\right)^2\right]$
(91) Rayleigh distribution	$F(x)=\begin{cases}0; & x\le b\\ 1-\mathrm{e}^{-(\frac{x-b}{a})^{2}}; & x>b\end{cases}$	$a>0$	$b+\dfrac{a\sqrt{\pi}}{2}$	$a^2(1-\dfrac{\pi}{4})$
(92) Chi-square distribution[6], $\chi^2(n)$, Gamma$(\frac{n}{2},\frac{1}{2})$	$f(x)=\begin{cases}0; & x\le 0\\ \frac{1/2}{\Gamma(\frac{n}{2})}\,(x/2)^{(n/2)-1}\,\mathrm{e}^{-x/2}; & x>0\end{cases}$ $F(x)=\begin{cases}0; & x\le 0\\ 1-\frac{\Gamma(\frac{n}{2};\frac{x}{2})}{\Gamma(\frac{n}{2})}; & x>0\end{cases}$	$n\in\mathsf{Z}_{\ge 1}$	n	$2n$
(93) Student's t-distribution[3, 7], t(n)	$f(x)=\frac{1}{\sqrt{n}}\cdot\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})}\cdot\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}}$ $F(x)=\begin{cases}\frac{1}{2}\cdot\frac{\mathrm{B}(\frac{n}{n+x^2};\frac{n}{2};\frac{1}{2})}{\mathrm{B}(1;\frac{n}{2};\frac{1}{2})}; & x<0\\ 1-\frac{1}{2}\cdot\frac{\mathrm{B}(\frac{n}{n+x^2};\frac{n}{2};\frac{1}{2})}{\mathrm{B}(1;\frac{n}{2};\frac{1}{2})}; & x\ge 0\end{cases}$	$n\in\mathsf{Z}_{\ge 1}$	0	$\dfrac{n}{n-2}$

[1] $\Gamma(a)$ is the gamma function; cf. (??). $\Gamma(a;b\cdot x)$ is the upper incomplete gamma function; cf. (??).

[2] γ is Euler's constant. $\gamma=\lim_{k\to\infty}\left((\sum_{i=1}^{k}\frac{1}{k})-\ln k\right)=0{,}577\ 215\ 664\ldots$

[3] Γ is the gamma funciton; cf. (??).

[4] Expectation exists if and only if $c>1$. Variance exists if and only if $c>2$.

[5] If X is a type III extreme value distributed random variable, then $-X$ (i.e. the negative of X) is Weibull distributed. Therefore the type III extreme value distribution now and then is called the extreme value distribution of Weibull type.

[6] $\Gamma(\frac{n}{2})$ is the gamma function; cf. (??). $\Gamma(\frac{n}{2};\frac{x}{2})$ is the upper incomplete gamma function; cf. (??).

[7] B is the incomplete beta function; cf. (??). Variance exists if and only if $n\ge 3$.

Distribution		Parameter restrictions	Expectation	Variance
(94) Fisher's F-distribution[1], $\mathrm{F}(n_1, n_2)$	$f(x) = \begin{cases} [x \le 0] = 0 \\ [x > 0] = \end{cases}$ $= \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\,\Gamma(\frac{n_2}{2})} \cdot \frac{{n_1}^{n_1/2}\,{n_2}^{n_2/2}\,x^{\frac{n_1}{2}-1}}{(n_2+n_1 x)^{(n_1+n_2)/2}}$ $F(x) = \begin{cases} 0; & x \le 0 \\ 1 - \frac{\mathrm{B}(\frac{n_2}{n_2+n_1 x};\frac{n_2}{2};\frac{n_1}{2})}{\mathrm{B}(1;\frac{n_2}{2};\frac{n_1}{2})}; & x > 0 \end{cases}$	$n_1 \in \mathsf{Z}_{\ge 1}$ $n_2 \in \mathsf{Z}_{\ge 1}$	$\frac{n_2}{n_2 - 2}$	$\frac{2{n_2}^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$
(95) Pareto distribution, $(c > 0)$	$F(x) = \begin{cases} 0; & x \le 0 \\ 1 - (1 - \frac{c \cdot x}{a})^{1/c}; & 0 < x < \frac{a}{c} \\ 1; & x \ge \frac{a}{c} \end{cases}$	$a > 0,\ c > 0$	$\frac{a}{c+1}$	$\frac{a^2}{(2c+1)(c+1)^2}$
(96) Pareto distribution[2], $(c < 0)$	$F(x) = \begin{cases} 0; & x \le 0 \\ 1 - (1 + \frac{\lvert c\rvert \cdot x}{a})^{-1/\lvert c\rvert}; & x > 0 \end{cases}$	$a > 0,\ c < 0$	$\frac{a}{c+1}$	$\frac{a^2}{(2c+1)(c+1)^2}$
(97) Pareto distribution[3], $(c = 0)$	$F(x) = \begin{cases} 0; & x \le 0 \\ 1 - \mathrm{e}^{-x/a}; & x > 0 \end{cases}$	$a > 0$	a	a^2

[1] B(.) is the incomplete beta function; cf. (??). Γ(.) is the gamma function; cf. (??). Expectation exists if and only if $n_2 \ge 3$. Variance exists if and only if $n_2 \ge 5$.

[2] Expectation exists if and only if $c > -1$. Variance exists if and only if $c > -\frac{1}{2}$.

[3] This is an exponential distribution, $\mathrm{Exp}(a)$.

Some functions

(98) The gamma function is defined (for $p>0$) by

(98.1) $$\Gamma(p) = \int_0^\infty \xi^{p-1} \mathrm{e}^{-\xi}\, \mathrm{d}\xi, \quad p>0$$

Some properties of the gamma function:

(98.2) $$\Gamma(p) = (p-1)!, \quad p \in \{1; 2; 3; \ldots\}$$

(98.3) $$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

(98.4) $$\Gamma(p+1) = p\,\Gamma(p); \quad p>0$$

(99) The incomplete gamma function is defined (for $p>0$, $x\ge 0$) by

(99.1) $$\Gamma(p; x) = \int_x^\infty \xi^{p-1} \mathrm{e}^{-\xi}\, \mathrm{d}\xi, \quad x\ge 0,\ p>0$$

A property of the incomplete gamma function:

(99.2) $$\Gamma(p; 0) = \Gamma(p), \quad p > 0$$

(100) The incomplete beta function is defined (for $a>0$, $b>0$, $0\le x\le 1$) by

(100.1) $$\mathrm{B}(x; a; b) = \int_0^x \xi^{a-1}(1-\xi)^{b-1}\, \mathrm{d}\xi, \quad 0\le x\le 1, \ a>0, \ b>0$$

A property of the incomplete beta function:

$$\text{(100.2)} \quad \mathrm{B}(1; a; b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \quad a>0, \;\; b>0$$

Table of the standard-normal distribution function

(101) If $X \in \mathrm{N}(0;1)$, then $\mathsf{P}(X \leq x) = \Phi(x)$, where $\Phi(\cdot)$ is a non-elementary function given by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\xi^2}{2}}\, \mathrm{d}\xi.$$

This table gives (with 5 correct decimals) the function values $\Phi(x)$ for $x = 0{,}00{:}0{,}01{:}3{,}99$. For negative values of x, use that $\Phi(-x) = 1 - \Phi(x)$.

—	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,500 00	0,503 99	0,507 98	0,511 97	0,515 95	0,519 94	0,523 92	0,527 90	0,531 88	0,535 86
0,1	0,539 83	0,543 80	0,547 76	0,551 72	0,555 67	0,559 62	0,563 56	0,567 49	0,571 42	0,575 35
0,2	0,579 26	0,583 17	0,587 06	0,590 95	0,594 83	0,598 71	0,602 57	0,606 42	0,610 26	0,614 09
0,3	0,617 91	0,621 72	0,625 52	0,629 30	0,633 07	0,636 83	0,640 58	0,644 31	0,648 03	0,651 73
0,4	0,655 42	0,659 10	0,662 76	0,666 40	0,670 03	0,673 64	0,677 24	0,680 82	0,684 39	0,687 93
0,5	0,691 46	0,694 97	0,698 47	0,701 94	0,705 40	0,708 84	0,712 26	0,715 66	0,719 04	0,722 40
0,6	0,725 75	0,729 07	0,732 37	0,735 65	0,738 91	0,742 15	0,745 37	0,748 57	0,751 75	0,754 90
0,7	0,758 04	0,761 15	0,764 24	0,767 30	0,770 35	0,773 37	0,776 37	0,779 35	0,782 30	0,785 24
0,8	0,788 14	0,791 03	0,793 89	0,796 73	0,799 55	0,802 34	0,805 11	0,807 85	0,810 57	0,813 27
0,9	0,815 94	0,818 59	0,821 21	0,823 81	0,826 39	0,828 94	0,831 47	0,833 98	0,836 46	0,838 91
1,0	0,841 34	0,843 75	0,846 14	0,848 49	0,850 83	0,853 14	0,855 43	0,857 69	0,859 93	0,862 14
1,1	0,864 33	0,866 50	0,868 64	0,870 76	0,872 86	0,874 93	0,876 98	0,879 00	0,881 00	0,882 98
1,2	0,884 93	0,886 86	0,888 77	0,890 65	0,892 51	0,894 35	0,896 17	0,897 96	0,899 73	0,901 47
1,3	0,903 20	0,904 90	0,906 58	0,908 24	0,909 88	0,911 49	0,913 09	0,914 66	0,916 21	0,917 74
1,4	0,919 24	0,920 73	0,922 20	0,923 64	0,925 07	0,926 47	0,927 85	0,929 22	0,930 56	0,931 89
1,5	0,933 19	0,934 48	0,935 74	0,936 99	0,938 22	0,939 43	0,940 62	0,941 79	0,942 95	0,944 08
1,6	0,945 20	0,946 30	0,947 38	0,948 45	0,949 50	0,950 53	0,951 54	0,952 54	0,953 52	0,954 49
1,7	0,955 43	0,956 37	0,957 28	0,958 18	0,959 07	0,959 94	0,960 80	0,961 64	0,962 46	0,963 27
1,8	0,964 07	0,964 85	0,965 62	0,966 38	0,967 12	0,967 84	0,968 56	0,969 26	0,969 95	0,970 62
1,9	0,971 28	0,971 93	0,972 57	0,973 20	0,973 81	0,974 41	0,975 00	0,975 58	0,976 15	0,976 70
2,0	0,977 25	0,977 78	0,978 31	0,978 82	0,979 32	0,979 82	0,980 30	0,980 77	0,981 24	0,981 69
2,1	0,982 14	0,982 57	0,983 00	0,983 41	0,983 82	0,984 22	0,984 61	0,985 00	0,985 37	0,985 74
2,2	0,986 10	0,986 45	0,986 79	0,987 13	0,987 45	0,987 78	0,988 09	0,988 40	0,988 70	0,988 99
2,3	0,989 28	0,989 56	0,989 83	0,990 10	0,990 36	0,990 61	0,990 86	0,991 11	0,991 34	0,991 58
2,4	0,991 80	0,992 02	0,992 24	0,992 45	0,992 66	0,992 86	0,993 05	0,993 24	0,993 43	0,993 61
2,5	0,993 79	0,993 96	0,994 13	0,994 30	0,994 46	0,994 61	0,994 77	0,994 92	0,995 06	0,995 20
2,6	0,995 34	0,995 47	0,995 60	0,995 73	0,995 85	0,995 98	0,996 09	0,996 21	0,996 32	0,996 43
2,7	0,996 53	0,996 64	0,996 74	0,996 83	0,996 93	0,997 02	0,997 11	0,997 20	0,997 28	0,997 36
2,8	0,997 44	0,997 52	0,997 60	0,997 67	0,997 74	0,997 81	0,997 88	0,997 95	0,998 01	0,998 07
2,9	0,998 13	0,998 19	0,998 25	0,998 31	0,998 36	0,998 41	0,998 46	0,998 51	0,998 56	0,998 61
3,0	0,998 65	0,998 69	0,998 74	0,998 78	0,998 82	0,998 86	0,998 89	0,998 93	0,998 96	0,999 00
3,1	0,999 03	0,999 06	0,999 10	0,999 13	0,999 16	0,999 18	0,999 21	0,999 24	0,999 26	0,999 29
3,2	0,999 31	0,999 34	0,999 36	0,999 38	0,999 40	0,999 42	0,999 44	0,999 46	0,999 48	0,999 50
3,3	0,999 52	0,999 53	0,999 55	0,999 57	0,999 58	0,999 60	0,999 61	0,999 62	0,999 64	0,999 65
3,4	0,999 66	0,999 68	0,999 69	0,999 70	0,999 71	0,999 72	0,999 73	0,999 74	0,999 75	0,999 76
3,5	0,999 77	0,999 78	0,999 78	0,999 79	0,999 80	0,999 81	0,999 81	0,999 82	0,999 83	0,999 83
3,6	0,999 84	0,999 85	0,999 85	0,999 86	0,999 86	0,999 87	0,999 87	0,999 88	0,999 88	0,999 89
3,7	0,999 89	0,999 90	0,999 90	0,999 90	0,999 91	0,999 91	0,999 92	0,999 92	0,999 92	0,999 92
3,8	0,999 93	0,999 93	0,999 93	0,999 94	0,999 94	0,999 94	0,999 94	0,999 95	0,999 95	0,999 95
3,9	0,999 95	0,999 95	0,999 96	0,999 96	0,999 96	0,999 96	0,999 96	0,999 96	0,999 97	0,999 97

Table of quantiles of Student's t-distribution

(102) If $X \in \mathrm{t}(n)$, then the α-quantile $t_\alpha(n)$ is defined by

$$\mathsf{P}\big(X > t_\alpha(n)\big) = \alpha, \quad 0 < \alpha < 1$$

This table gives (with 3 correct decimals) the α-quantile $t_\alpha(n)$ for $\alpha \in \{0{,}1; 0{,}05; 0{,}025; 0{,}01; 0{,}005; 0{,}001; 0{,}0005\}$ and for $n \in \{1{:}1{:}30; 40; 60; 120\}$. For values of $\alpha \geq 0{,}9$, use that

$$t_{1-\alpha}(n) = -t_\alpha(n), \quad 0 < \alpha < 1$$

n	α						
	0,1	0,05	0,025	0,01	0,005	0,001	0,0005
1	3,078	6,314	12,706	31,821	63,657	318,309	636,619
2	1,886	2,920	4,303	6,965	9,925	22,327	31,599
3	1,638	2,353	3,182	4,541	5,841	10,215	12,924
4	1,533	2,132	2,776	3,747	4,604	7,173	8,610
5	1,476	2,015	2,571	3,365	4,032	5,893	6,869
6	1,440	1,943	2,447	3,143	3,707	5,208	5,959
7	1,415	1,895	2,365	2,998	3,499	4,785	5,408
8	1,397	1,860	2,306	2,896	3,355	4,501	5,041
9	1,383	1,833	2,262	2,821	3,250	4,297	4,781
10	1,372	1,812	2,228	2,764	3,169	4,144	4,587
11	1,363	1,796	2,201	2,718	3,106	4,025	4,437
12	1,356	1,782	2,179	2,681	3,055	3,930	4,318
13	1,350	1,771	2,160	2,650	3,012	3,852	4,221
14	1,345	1,761	2,145	2,624	2,977	3,787	4,140
15	1,341	1,753	2,131	2,602	2,947	3,733	4,073
16	1,337	1,746	2,120	2,583	2,921	3,686	4,015
17	1,333	1,740	2,110	2,567	2,898	3,646	3,965
18	1,330	1,734	2,101	2,552	2,878	3,610	3,922
19	1,328	1,729	2,093	2,539	2,861	3,579	3,883
20	1,325	1,725	2,086	2,528	2,845	3,552	3,850
21	1,323	1,721	2,080	2,518	2,831	3,527	3,819
22	1,321	1,717	2,074	2,508	2,819	3,505	3,792
23	1,319	1,714	2,069	2,500	2,807	3,485	3,768
24	1,318	1,711	2,064	2,492	2,797	3,467	3,745
25	1,316	1,708	2,060	2,485	2,787	3,450	3,725
26	1,315	1,706	2,056	2,479	2,779	3,435	3,707
27	1,314	1,703	2,052	2,473	2,771	3,421	3,690
28	1,313	1,701	2,048	2,467	2,763	3,408	3,674
29	1,311	1,699	2,045	2,462	2,756	3,396	3,659
30	1,310	1,697	2,042	2,457	2,750	3,385	3,646
40	1,303	1,684	2,021	2,423	2,704	3,307	3,551
60	1,296	1,671	2,000	2,390	2,660	3,232	3,460
120	1,289	1,658	1,980	2,358	2,617	3,160	3,373
∞	1,282	1,645	1,960	2,326	2,576	3,090	3,291

Table of quantiles of the χ^2 distribution

(103) If $X \in \chi^2(n)$, then the α-quantile $\chi^2_\alpha(n)$ is defined by

$$\mathsf{P}\big(X > \chi^2_\alpha(n)\big) = \alpha, \quad 0 < \alpha < 1$$

This table gives the α-quantile $\chi^2_\alpha(n)$ for $\alpha \in \{0{,}9995; 0{,}999; 0{,}99; 0{,}975; 0{,}95; 0{,}05; 0{,}025; 0{,}01; 0{,}005; 0{,}001; 0{,}0005\}$ and for $n \in \{1{:}1{:}30; 40{:}10{:}100\}$.

n	α										
	0,9995	0,999	0,995	0,99	0,975	0,95	0,05	0,025	0,01	0,005	0,001	0,0005
1	—	—	$< 10^{-2}$	$< 10^{-2}$	$< 10^{-2}$	$< 10^{-2}$	3,841	5,024	6,635	7,879	10,83	12,12
2	$< 10^{-2}$	$< 10^{-2}$	0,0100	0,0201	0,0506	0,1026	5,991	7,378	9,210	10,60	13,82	15,20
3	0,0153	0,0240	0,0717	0,1148	0,2158	0,3518	7,815	9,348	11,34	12,84	16,27	17,73
4	0,0639	0,0908	0,2070	0,2971	0,4844	0,7107	9,488	11,14	13,28	14,86	18,47	20,00
5	0,1581	0,2102	0,4117	0,5543	0,8312	1,145	11,07	12,83	15,09	16,75	20,52	22,11
6	0,2994	0,3811	0,6757	0,8721	1,237	1,635	12,59	14,45	16,81	18,55	22,46	24,10
7	0,4849	0,5985	0,9893	1,239	1,690	2,167	14,07	16,01	18,48	20,28	24,32	26,02
8	0,7104	0,8571	1,344	1,646	2,180	2,733	15,51	17,53	20,09	21,95	26,12	27,87
9	0,9717	1,152	1,735	2,088	2,700	3,325	16,92	19,02	21,67	23,59	27,88	29,67
10	1,265	1,479	2,156	2,558	3,247	3,940	18,31	20,48	23,21	25,19	29,59	31,42
11	1,587	1,834	2,603	3,053	3,816	4,575	19,68	21,92	24,72	26,76	31,26	33,14
12	1,934	2,214	3,074	3,571	4,404	5,226	21,03	23,34	26,22	28,30	32,91	34,82
13	2,305	2,617	3,565	4,107	5,009	5,892	22,36	24,74	27,69	29,82	34,53	36,48
14	2,697	3,041	4,075	4,660	5,629	6,571	23,68	26,12	29,14	31,32	36,12	38,11
15	3,108	3,483	4,601	5,229	6,262	7,261	25,00	27,49	30,58	32,80	37,70	39,72
16	3,536	3,942	5,142	5,812	6,908	7,962	26,30	28,85	32,00	34,27	39,25	41,31
17	3,980	4,416	5,697	6,408	7,564	8,672	27,59	30,19	33,41	35,72	40,79	42,88
18	4,439	4,905	6,265	7,015	8,231	9,390	28,87	31,53	34,81	37,16	42,31	44,43
19	4,912	5,407	6,844	7,633	8,907	10,12	30,14	32,85	36,19	38,58	43,82	45,97
20	5,398	5,921	7,434	8,260	9,591	10,85	31,41	34,17	37,57	40,00	45,31	47,50
21	5,896	6,447	8,034	8,897	10,28	11,59	32,67	35,48	38,93	41,40	46,80	49,01
22	6,404	6,983	8,643	9,542	10,98	12,34	33,92	36,78	40,29	42,80	48,27	50,51
23	6,924	7,529	9,260	10,20	11,69	13,09	35,17	38,08	41,64	44,18	49,73	52,00
24	7,453	8,085	9,886	10,86	12,40	13,85	36,42	39,36	42,98	45,56	51,18	53,48
25	7,991	8,649	10,52	11,52	13,12	14,61	37,65	40,65	44,31	46,93	52,62	54,95
26	8,538	9,222	11,16	12,20	13,84	15,38	38,89	41,92	45,64	48,29	54,05	56,41
27	9,093	9,803	11,81	12,88	14,57	16,15	40,11	43,19	46,96	49,64	55,48	57,86
28	9,656	10,39	12,46	13,56	15,31	16,93	41,34	44,46	48,28	50,99	56,89	59,30
29	10,23	10,99	13,12	14,26	16,05	17,71	42,56	45,72	49,59	52,34	58,30	60,73
30	10,80	11,59	13,79	14,95	16,79	18,49	43,77	46,98	50,89	53,67	59,70	62,16
40	16,91	17,92	20,71	22,16	24,43	26,51	55,76	59,34	63,69	66,77	73,40	76,09
50	23,46	24,67	27,99	29,71	32,36	34,76	67,50	71,42	76,15	79,49	86,66	89,56
60	30,34	31,74	35,53	37,48	40,48	43,19	79,08	83,30	88,38	91,95	99,61	102,7
70	37,47	39,04	43,28	45,44	48,76	51,74	90,53	95,02	100,4	104,2	112,3	115,6
80	44,79	46,52	51,17	53,54	57,15	60,39	101,9	106,6	112,3	116,3	124,8	128,3
90	52,28	54,16	59,20	61,75	65,65	69,13	113,1	118,1	124,1	128,3	137,2	140,8
100	59,90	61,92	67,33	70,06	74,22	77,93	124,3	129,6	135,8	140,2	149,4	153,2