

# Lecture 1. Probabilities - Definitions, Examples and Basic Tools

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**Uncertain World:** *How will global warming affect the planet? When and where will earthquakes and other natural disasters strike? ... The public is often puzzled when scientists cant answer these questions with certainty, and wonder if there will ever be enough scientific information to make sound decisions for the future. Example*

**Risk** *is a quantity derived both from the probability that a particular hazard will occur and the magnitude of the consequence of the undesirable effects of that hazard. The term risk is often used informally to mean the probability of a hazard occurring.*

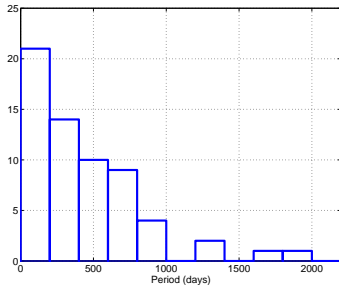
**Probabilities** *are numbers, assigned to (events) statements about outcome of an experiment, that express the chances that the statement is true.*

**Statistics** *is the scientific application of mathematical principles to the collection, analysis, and presentation of numerical data.*

## Common usages of the concept of probability:

- ▶ To describe variability of outcomes of repeatable experiments, e.g. chances of getting “Heads” in a flip of a coin, chances of failure of a component (mass production) or of occurrence of a large earthquake worldwide during one year. In the situation when the number of repetitions of the experiment is uncertain, e.g. the probability of fire after lightning has hit a building.
- ▶ To quantify the uncertainty of an outcome of a non-repeatable event. Here the probability will depend on the available information.  
**How many years can the ship be safely used?**
- ▶ To measure the present state of knowledge, e.g. the probability that the detected tumor is malignant.

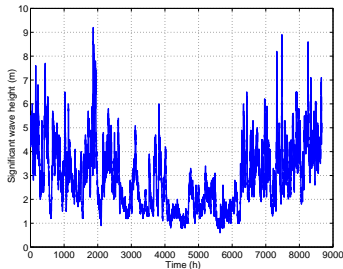
# Examples of data



Histogram: Periods in days between serious earthquakes 1902–1977.

Probabilistic questions?

Probabilistic models?



Measurements of Significant wave height,

Jan 1995 – Dec 1995 100-years significant wave?

# Probabilities

Term *experiment* is used to refer to any process whose outcome is not known in advance. Consider an experiment.

- ▶ Sample space  $\mathcal{S}$ : A collection of all possible outcomes.
- ▶ Sample point  $s \in \mathcal{S}$ : An element in  $\mathcal{S}$ .
- ▶ Event  $A$ : A subset of sample points,  $A \subset \mathcal{S}$  for which a statement about an outcome is true.

## Rules for probabilities:

$$P(A \cup B) = P(A) + P(B), \quad \text{if } A \cap B = \emptyset.$$

For any event  $A$ ,

$$0 \leq P(A) \leq 1.$$

Statements which are always false have **probability zero**, similarly,

always-true statements have **probability one**.

Example 1

## Probabilities - Kolmogorov's axioms.

**Definition.** Let  $A_1, A_2, \dots$  be an infinite sequence of statements such that at most one of them can be true ( $A_i$  are mutually excluding); then

$$P(\text{"At least one of } A_i \text{ is true"}) = P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i). \quad (*)$$

Any function  $P$  satisfying (\*) taking values between zero and one and assigning

- ▶ value zero to never-true statements (impossible events)
- ▶ value one to always-true statements (certain events)

**is a correctly defined probability.**

*Example 2*

# A.N. Kolmogorov



A.N. Kolmogorov (1903-1987).

ERGEBNISSE DER MATHEMATIK  
UND IHRER GRENZGEBIETE  
HERAUSGEGEBEN VON DER SCHRIFTLICHTUNG  
DES  
„ZENTRALBLATT FÜR MATHEMATIK“  
ZWEITER BAND

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GRUNDBEGRIFFE DER  
WAHRSCHEINLICHS-  
RECHNUNG

VON  
A. KOLMOGOROFF



BERLIN  
VERLAG VON JULIUS SPRINGER  
1933

Grundbegriffe der  
Wahrscheinlichkeitsrechnung.

## How to find "useful" probabilities:

- ▶ Classically for finite sample spaces  $\mathcal{S}$ , if all outcomes are equally probable then

$$P(A) = \text{number of outcomes for which } A \text{ is true} / \text{number of outcomes}$$

- ▶ Employ a concept of independence.
- ▶ Employ a concept of conditional probabilities.

If everybody agrees with the choice of  $P$ , it is called an **objective probability**. (If a coin is "fair" the probability of getting tails is 0.5.)

For many problems the probability will depend on the information a person has when estimating the chances that a statement  $A$  is true. One then speaks of **subjective probability**.



# Independence

For a sample space  $\mathcal{S}$  and a probability measure  $P$ , the events  $A, B \subset \mathcal{S}$  are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Two events  $A$  and  $B$  are **dependent** if they are not independent, i.e.

$$P(A \cap B) \neq P(A) \cdot P(B).$$

*Example 3*

# Conditional probability

Conditional probability:

$$P(B | A) = \frac{P(A \cap B)}{P(A)}$$

The chances that some statement  $B$  is true when we *know* that some statement  $A$  is true.

*Example 4*

## Law of total probability

Let  $A_1, \dots, A_n$  be a partition of the sample space.

Then for any event  $B$

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_n)$$

*Example 5*

## Bayes' formula

Again let  $A_1, \dots, A_n$  be a partition of the sample space, i.e. we have  $n$  excluding hypothesis and only one of them is true. The evidence is that  $B$  is true. Which of alternatives is most likely to be true?

Bayes' formula:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(A_i)}{P(B)}P(B|A_i)$$

Name due to *Thomas Bayes* (1702-1761)

**Likelihood:**  $L(A_i) = P(B|A_i)$  (How likely is the observed event  $B$  under alternative  $A_i$ ?)

**Example 6**

**Odds**<sup>1</sup> for events  $A_1$  and  $A_2$ : Any positive numbers  $q_1$  and  $q_2$  such that

$$\frac{q_1}{q_2} = \frac{P(A_1)}{P(A_2)}$$

Let  $A_1, A_2, \dots, A_n$  be a partition of the sample space having odds  $q_i$ , i.e.  $P(A_j)/P(A_i) = q_j/q_i$ . Then

$$P(A_i) = \frac{q_i}{q_1 + \dots + q_n}$$

The language of odds such as "ten to one" for intuitively estimated risks is found in the sixteenth century, well before the invention of mathematical probability.[1] Shakespeare writes in Henry IV:

*Knew that we ventured on such dangerous seas that if we wrought out life 'twas ten to one.*

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<sup>1</sup>European odds:  $q_i = 1/P(A_i)$  - if you bet 1 SEK on  $A_i$  then you get  $q_i$  SEK if  $A_i$  is true (occurs) or loos your bet if  $A_i$  is false.

## Bayes' formula formulated using odds

Bayes' formula can be conveniently written by means of *odds*:

$$q_i^{\text{post}} = P(B | A_i) q_i^{\text{prior}}$$

Prior odds:  $q_i^{\text{prior}}$  for  $A_i$  **before**  $B$  is known.

Posterior odds:  $q_i^{\text{post}}$  for  $A_i$  **after** it is known that  $B$  is true.

*Example 7*

## Conditional independence

Events  $B_1, B_2$  are **conditionally independent** given a partition  $A_1, A_2, \dots, A_k$  of the sample space if

$$P(B_1 \cap B_2 | A_i) = P(B_1 | A_i)P(B_2 | A_i).$$

Events  $B_1, \dots, B_n$  are conditionally independent if all pairs  $B_i, B_j$  are conditionally independent.

## Odds: Recursive updating

Let  $A_1, A_2, \dots, A_k$  be a partition of the sample space, and  $B_1, \dots, B_n, \dots$  a sequence of true statements (evidences).

If the evidences  $B$  are conditionally independent of  $A_i$  then the **posterior odds** can be computed recursively using the induction

$$q_i^0 = q_i^{\text{prior}}$$

$$q_i^n = P(B_n | A_i) q_i^{n-1}, \quad n = 1, 2, \dots$$

*Example 8: Waste-water treatment:*



## Check your intuition:

Are independent events  $B_1, B_2$  always conditionally independent?  
see [for help "click"](#) .

Solve problem 2.6.

[Examples in this lecture "click"](#)