Chalmers-University of Gothenburg Department of Mathematical Sciences

Probability, Statistics and Risk, MVE240

Project 8

In the project description we sketch the analysis of the problem we expect you to do. (Obviously you are welcome to do more.) To pass the project a short report should be written and handed in to the project supervisor. In addition the group should present their results in class. The presentation should take about 15 minutes. Please include a short introduction which will facilitate for other students to understand the results of the project. (Do not assume that the audience knows the subject.)

1 Introduction - spatial Poisson processes

You have previously been acquainted with homogenous Poisson point processes in time, Section 2.6.1 where they were referred to as Poisson streams. In Section 7.5, (homogenous) Poisson point processes in space, also called spatial Poisson processes, were introduced. In this project, we will deal with simulation issues for spatial Poisson processes, after a short survey of these processes. You might want to review Section 7.5 before you continue.
The important characteristic of a Poisson process in time is that the expected number of events in a time interval relates to the (possibly time-dependent) intensity of the process. The spatial Poisson process is a generalization of such processes, where events are no longer events in time but points in some set $W \subset \mathbf{R}^n$. Consider for simplicity a set $W = [0,1] \times [0,1] \subset \mathbf{R}^2$. A homogeneous spatial Poisson process is then the realization of a stochastic number of points (vectors), placed randomly in W, independent of each other, according to some distribution. Moreover, sampling this spatial Poisson process is equivalent to sampling a Poisson process with some known rate function, and associating with each event a random vector $(x, y) \in W$ sampled from some probability density function. Hence, there actually is a time dimension to this problem in a sense, but we shall not be concerned with aspects of when events occur, only how many and the spatial vectors associated with them.
As a matter of fact, one may skip the notion of time in the definition and define a homogeneous planar Poisson process by the following conditions:

- For some $\lambda > 0$, and any finite planar region A, $N(A)$ (the number of events with corresponding vectors in A) follows a Poisson distribution with mean $\lambda|A|$ (where $|A|$ is the area of A). We here call λ the intensity of the process.
- Given $N(A) = n$, the n events in A form an independent random sample from the uniform distribution on A.

Note that $\lambda|A|$ is only the integral of λ over A. To get an inhomogeneous planar Poisson process, one need now only replace the constant λ by a spatially dependent intensity $\lambda(x, y)$. The first condition above shall then be replaced by

- For some $\lambda(x, y) > 0$, and any finite planar region A, $N(A)$ (the number of events with corresponding vectors in A) follows a Poisson distribution with mean $\int_A \lambda(x, y)dx\,dy$.

The expected number of events in total should hence be $\lambda = \int_W \lambda(x, y)dx\,dy$, and the distribution according to which vectors in W are distributed is proportional (why is this?) to $\lambda(x, y)$, $f(x, y) = \lambda(x, y)/\lambda$.

2 Sampling from a homogeneous spatial Poisson process

This chapter can be considered a warm-up. Consider a homogeneous spatial Poisson process with events in $W = [0,1] \times [0,1]$. The process is from now on described by the (homogeneous) intensity $\lambda > 0$, and the expected number of events is $\lambda|W| = \lambda$. Type the following lines of code:

```
>> lambda = 5;
>> sizeW = 1;
>> noEvents = poissrnd(lambda*sizeW);
>> coords = rand(noEvents,2);
>> figure(1), hold on
>> axis([0 1 0 1]);
>> plot(coords(:,1),coords(:,2),'.')
```

You now see one realization of this spatial Poisson process. Both the number of points and the coordinates are random. We now move on to a more difficult case.

3 Sampling from an inhomogeneous spatial Poisson process

3.1 Method I

The case of an inhomogeneous spatial Poisson process affects the way the matrix `coords` above should be sampled. Let the intensity be $\lambda(x, y) = ke^x e^{-y}$ for some $k > 0$. What is then the corresponding $f(x, y)$? How do you sample from this distribution? What property of $f(x, y)$ and the relation between the corresponding stochastic variables X and Y are you using?

3.2 Method II

Another method of sampling from this distribution is to uniformly randomize a vector (x, y) in W and then rejecting this vector with probability $1 - \lambda(x, y)/\lambda^*$, where λ^* is the maximum of λ for all x and y. This is repeated until the number of accepted samples reaches an already determined number (determined by randomization). Implement this method and convince yourself that the two methods seem to give the same distribution (try for example $k = 1000$ or $k = 5000$).

3.3 A comparison of the methods

Would there be a way to test this hypothesis that the two algorithms actually sample from the same distribution? If they give the same results, would you prefer one over the other? Quite clearly, most would experience method I as the more intuitive. But suppose you have a situation where X and Y are dependent in some complex manner. Which method would then be the easiest to implement? You are of course very welcome to come up with a non separable intensity $\lambda(x, y)$ and try to repeat the simulations above.