Solutions: Written examination - 19 August 2009
Sannolikhet, statistik och risk MVE240.

1 We search the 50-years snow load s_{50}. It satisfies $F(s_{50}) = 1 - 1/50 = 0.98$, thus $\exp(-(\exp(-\frac{s_{50}-13}{1.5}))) = 0.98$. Solving for s_{50} gives

$$s_{50} = 13 - 1.5 \cdot \ln(-\ln(0.98)) = 18.9.$$

2 Denote the lifetime by the stochastic variable T. The searched probability is $P(T > 15000) = \exp(-\int_0^{15000} \lambda(s)ds) = \exp(-\int_0^{15000} 5 \cdot 10^{-7} + 3 \cdot 10^{-13} \cdot s^2 ds) = \exp(-5 \cdot 10^{-7} \cdot 15000 - 10^{-13} \cdot 15000^3) = 0.71.$

3 From table: $c = 5.73$. Furtermore, $F_X(x_{0.9}) = 1 - e^{-(x/a)^c} = 0.1 \Rightarrow a = \frac{x_{0.9}}{(-\ln 0.9)^{1/c}} = 14.81$. Thus the searched probability is, $P(X < 8) = F_X(8) = 1 - e^{-(8/14.81)^{5.73}} = 0.029$

4 From the hint we have that t_m is equal to 0.9-quantile of T, i.e. it is a solution to the equation

$$\mathsf{P}(T \le t_m) = 1 - 0.9.$$

Since T is lognormally distributed then one needs to find $m_T = \mathsf{E}[\ln(T)]$ and $\sigma_T^2 = \mathsf{V}[\ln(T)]$. Those are computed using the relations

$$\sigma_T^2 = \ln(1 + \mathsf{R}(T)^2), \quad \mathsf{E}[T] = \exp(m_T + \sigma_T^2/2).$$

Since $\mathsf{R}(T) = 0.25$ we have that $\sigma_T^2 = 0.06$ and $m_T = 1.7618$. Now

$$1 - 0.9 = \mathsf{P}(T \le t_m) = \Phi\left(\frac{\ln(t_m) - m_T}{\sigma_T}\right)$$

and hence

$$t_m = e^{1.7618 - 1.28\sqrt{0.06}} = 4.25 \quad \text{[moth]}.$$

5 **(a)** The number of accidents in one month is denoted N. The accidents is assumed to follow the Poisson distribution. The intensity of accidents Λ is uncertain and we model it as a random variable. Given the intensity, the probability of at least one accident in the time interval $[0, t]$,

$$P(\text{'at least one accident'}|\Lambda = \lambda) = 1 - P(\text{'No accidents'}|\Lambda = \lambda) = 1 - \exp(-\lambda t).$$

The prior distribution of λ is set to an improper gammafunction, $\Gamma(0,0)$. Using the table of formulæ we obtain the posterior distribution $\Gamma(6,4)$, since 6 is the total number of accidents and 4 the number of years. By the law of total probability the searched probability is obtained, using $t = 1/12$,

$$P(N \geq 1) = 1 - P(N = 0)$$
$$= 1 - \int_0^\infty \exp(-\lambda \cdot \frac{1}{12}) \frac{4^6}{\Gamma(4)} \lambda^5 e^{-4\lambda} d\lambda = 1 - \left(\frac{4}{1/12 + 4}\right)^6 = 0.12.$$

(b) Denote the number of accidents with trucks that really has dangerous goods with M. Similarly to exercise 5 (a), the probability that at least one accident occur during one month is,

$$P(M \geq 1 | \Lambda = \lambda) = 1 - \exp(-\lambda P(B) t),$$

where P(B)=0.5. Again, by the law of total probability we have,

$$P(M \geq 1) = 1 - \int_0^\infty \exp(-P(B)\lambda \cdot \frac{1}{12}) \frac{4^6}{\Gamma(4)} \lambda^5 e^{-4\lambda} d\lambda$$
$$= 1 - \left(\frac{4}{P(B)\frac{1}{12} + 4}\right)^6 = 0.06.$$

6 **(a)** $\lambda^* = \frac{65}{20000} \mathrm{m}^{-2}$.

(b) $m^* = \lambda^* \cdot \text{area} = \lambda^* \cdot \frac{20000}{25} = 2.6$

(c) Hypothesis: $H_0 : N \in \text{Po}(2.6)$. If H_0 is true then $P(N = i) = \exp(-2.6) \cdot \frac{2.6^i}{i!}$. We use the χ^2 test to test the hypothesis. The outcomes are grouped into 5 parts, $N = 0, N = 1, N = 2, N = 3, N \geq 4$, with corresponding probabilities, 0.0743, 0.1931, 0.2510, 0.2176, 0.2640. The number of outcomes in each group is 1, 4, 5, 11, 4.

$$Q = \sum_{i=1}^{5} \frac{(x_i - 25p_i)^2}{25p_i} = 7.5 < \chi^2_{0.05}(5 - 1 - 1).$$

Since the 5%-quantile of the $\chi^2(5 - 1 - 1)$ distribution is 7.81, which is greater than 7.5, we do not reject H_0.