Lecture 1. Probabilities - Definitions, Examples and Basic Tools

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Risk *is a quantity derived both from the probability that a particular hazard will occur and the magnitude of the consequence of the undesirable effects of that hazard. The term risk is often used informally to mean the probability of a hazard occurring.*

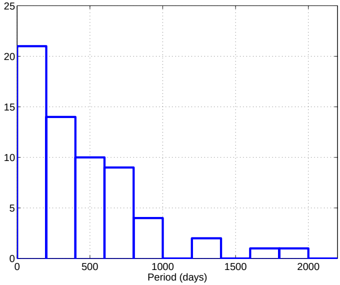
Probabilities *are numbers, assigned to (events) statements about outcome of an experiment, that express the chances that the statement is true.*

Statistics *is the scientific application of mathematical principles to the collection, analysis, and presentation of numerical data.*

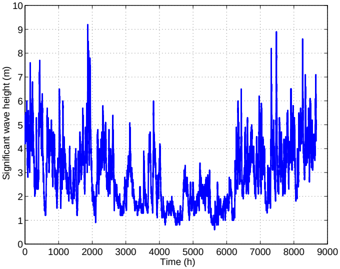
Common usages of the concept of probability:

- To describe variability of outcomes of repeatable experiments, e.g. chances of getting "Heads" in a flip of a coin, chances of failure of a component (mass production) or of occurrence of a large earthquake worldwide during one year.
- To quantify the uncertainty of an outcome of a non-repeatable event. Here the probability will depend on the available information. Example. (Modeling, estimation of model parameters, evaluations of probabilities, uncertainties of risks estimates.)
- To measure the present state of knowledge, e.g. the probability that the detected tumor is malignant.

Examples of data



Histogram: Periods in days between serious earthquakes 1902–1977.



Measurements of Significant wave height,

Jan 1995 – Dec 1995.

How high is 100-years significant wave?

Probabilities

Term *experiment* is used to refer to any process whose outcome is not known in advance. Consider an experiment.

- Sample space $\mathcal{S}$: A collection of all possible outcomes.
- Sample point $s \in \mathcal{S}$: An element in $\mathcal{S}$.
- Event A: A subset of sample points, $A \subset \mathcal{S}$ for which a statement about an outcome is true.

Rules for probabilities:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B), \quad \text{if} \quad A \cap B = \emptyset.$$

For any event A,

$$0 \leq \mathrm{P}(A) \leq 1.$$

Statements which are always false have **probability zero**, similarly,

always-true statements have **probability one** *Example 1*.

Independence

For a sample space $\mathcal{S}$ and a probability measure P, the events $A, B \subset \mathcal{S}$ are called **independent** if

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) \cdot \mathsf{P}(B).$$

Two events A and B are **dependent** if they are not independent, i.e.

$$\mathsf{P}(A \cap B) \neq \mathsf{P}(A) \cdot \mathsf{P}(B).$$

Example 2

How to find "useful" probabilities:

- Classically for finite sample spaces $\mathcal{S}$, if all outcomes are equally probable then

  $\mathrm{P}(A) =$ number of outcomes for which A is true/number of outcomes

- Employ a concept of independence. *Example 3*
- Employ a concept of conditional probabilities.

If everybody agrees with the choice of P, it is called an **objective probability**. (If a coin is "fair" the probability of getting tails is 0.5.)

For many problems the probability will depend on the information a person has when estimating the chances that a statement A is true. One then speaks of **subjective probability**.

Conditional probability

Conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

The chances that some statement B is true when we *know* that some statement A is true.

Example 4

Law of total probability

Let $A_1, \dots, A_n$ be a partition of the sample space.

Then for any event B

$$\mathrm{P}(B) = \mathrm{P}(B|A_1)\mathrm{P}(A_1) + \mathrm{P}(B|A_2)\mathrm{P}(A_2) + \cdots + \mathrm{P}(B|A_n)\mathrm{P}(A_n)$$

Example 5

Bayes' formula

Again let $A_1, \ldots, A_n$ be a partition of the sample space, i.e. we have n excluding hypothesis and only one of them is true. The evidence is that B is true. Which of alternatives is most likely to be true?

Bayes' formula:

$$\mathrm{P}(A_i|B) = \frac{\mathrm{P}(A_i \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(B|A_i)\mathrm{P}(A_i)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A_i)}{\mathrm{P}(B)}\mathrm{P}(B|A_i)$$

Name due to *Thomas Bayes* (1702-1761)

Likelihood: $L(A_i) = \mathrm{P}(B|A_i)$ (How likely is the observed event B under alternative A_i?)

$\boxed{\textit{Example 6}}$

Odds[1] (fractional odds) [2] for events A_1 and A_2: Any positive numbers q_1 and q_2 such that

$$\frac{q_1}{q_2} = \frac{\mathrm{P}(A_1)}{\mathrm{P}(A_2)}$$

Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space having odds q_i, i.e. $\mathrm{P}(A_j)/\mathrm{P}(A_i) = q_j/q_i$. Then

$$\mathrm{P}(A_i) = \frac{q_i}{q_1 + \cdots + q_n}$$

[1] The language of odds such as "ten to one" for intuitively estimated risks is found in the sixteenth century, well before the invention of mathematical probability.[1] Shakespeare writes in Henry IV:

Knew that we ventured on such dangerous seas that if we wrought out life 'twas ten to one.

[2] European odds: $q_i = 1/\mathrm{P}(A_i)$ - if you bet 1 SEK on A_i then you get q_i SEK if A_i is true (occurs) or loos your bet if A_i is false. Odds 1:4 corresponds to Europian odds 5.

Bayes' formula formulated using odds

Bayes' formula can be conveniently written by means of *odds*:

$$q_i^{\text{post}} = \mathrm{P}(B \mid A_i) q_i^{\text{prior}}$$

Prior odds: q_i^{prior} for A_i **before** B is known.

Posterior odds: q_i^{post} for A_i **after** it is known that B is true.

Example 7

Conditional independence

Events B_1, B_2 are **conditionally independent** given a partition $A_1, A_2, \ldots, A_k$ of the sample space if

$$\mathrm{P}(B_1 \cap B_2 | A_i) = \mathrm{P}(B_1 \mid A_i)\mathrm{P}(B_2 \mid A_i).$$

Events $B_1, \ldots, B_n$ are conditionally independent if all pairs B_i, B_j are conditionally independent.

Odds: Recursive updating

Let $A_1, A_2, \dots, A_k$ be a partition of the sample space, and $B_1, \dots, B_n, \dots$ a sequence of true statements (evidences).

If the evidences B are conditionally independent of A_i then the **posterior odds** can be computed recursively using the induction

$$q_i^0 = q_i^{\text{prior}}$$

$$q_i^n = \mathrm{P}(B_n \mid A_i) q_i^{n-1}, \qquad n = 1, 2, \dots$$

Example 8: Waste-water treatment:

Solve problem 2.6. Examples in this lecture "click"