PROBABILITY, STATISTICS AND RISK MVE300

Computer exercise 5
System Reliability

1   Preparatory exercises

1. Read the instructions for the computer exercise and chapter 8-8.1 in the textbook.

2. On page 201 in the textbook the distribution for the minimum of n independent variabels, $X_1, \ldots, X_n$, is computed.

Question 1: Give the distribution for the maximum, i.e. compute $\mathsf{P}(\max(X_1, \ldots, X_n) \leq z)$.

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3. Solve exercise 8.8 (without looking at the solution!). Also, give the *failure function* for this example.

4. Show that $\mathsf{P}(S \leq R) = \int_{-\infty}^{\infty} F_S(r) f_R(r) \, \mathrm{d}r$.

2   System Reliability

The *reliability* of an engineering system[1] is often defined as the probability that the system will function as intended. We will also refer to the opposite concept, namely the *failure probability* P_{f} (f stands for failure), which is the probability that the system will *not* function as intended. The level of performance of a system will obviously depend on the properties of the system.

Assume that all interesting properties of an engineering system are described by a set of parameters $x_1, x_2, \ldots, x_n$. We want the system to endure a set of loads of our choice[2] (the system might be subjected to more than one load). The magnitudes of these loads — let us denote them $y_1, y_2, \ldots, y_m$ — must however be limited, due to engineering imperfection, cost limits, time limits, and the like: we understand that there are combinations of $y_1, y_2, \ldots, y_m$ and $x_1, x_2, \ldots, x_n$ where the system capacity is exceeded and where the system will inevitably break down. We formalise this

$$
\begin{aligned}
\text{The system functions as intended} &\Leftrightarrow h(y_1; \ldots; y_m; x_1; \ldots; x_n) > 0 \\
\text{The system does not function as intended} &\Leftrightarrow h(y_1; \ldots; y_m; x_1; \ldots; x_n) < 0
\end{aligned}
$$

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[1] e.g. a construction, a vehicle, a production line, a multi-article stock-room logistic system, a computer network, a nuclear power-plant, a dam, a communication satellite, or a finance portfolio.

[2] e.g. the construction must bear a certain amount of wind load or weight; the vehicle must cover a satisfactory distance before its engine starts malfunctioning; the production line must produce goods continuously for at least a week (say) to be profitable; the stock-room logistic system must deliver at least 99 % (say) of the goods on order on time and to the right orderer; etc.

The function h is called the *failure function* (performance function, state function). If the parameters and the applied "loads" are marred by randomness, we instead treat them as random variables Y_1, Y_2, ..., Y_m and X_1, X_2, ..., X_n. In terms hereof, we can now write the failure probability P_f as

$$P_f = \mathsf{P}(h(Y_1; \ldots; Y_m; X_1; \ldots; X_n) < 0)$$

[3]In this computer exercise, our goal is to calculate P_f. The function h will always be given, as will the distribution functions of Y_1, ..., Y_m and X_1, ..., X_n. First we will obtain P_f from numerical integration *and* simulations, then from an explicit expression *and* from simulations, and eventually from simulations only. No real-world data today!

3   A standard example

Consider a construction of some kind. A load denoted S has the distribution function

$$F_S(x) = \exp(-\mathrm{e}^{-(x-b_S)/a_S})$$

where $b_S = 55$ and $a_S = 2,5$ and the construction has a strength (resistance) R with distribution

$$F_R(x) = \begin{cases} 1 - \exp(-((x - b_R)/a_R)^{c_R}), & x > b_R \\ 0, & x \le b_R \end{cases}$$

where $b_R = 70$, $a_R = 5$, and $c_R = 2$.

Question 2: Identify names of the distributions used to model S and R?

Assume that load and strength are independent. In this case, the failure function h is simple:

$$h(R; S) = R - S$$

We will first determine the probability of failure, $P_f = \mathsf{P}(h(R; S) < 0) = \mathsf{P}(R - S < 0) = \mathsf{P}(S > R)$. It can be shown that P_f can be written as

$$P_f = \mathsf{P}(S > R) = 1 - \mathsf{P}(S \le R) = 1 - \int_{-\infty}^{\infty} F_S(r) f_R(r) \, \mathrm{d}r =$$

$$= 1 - \int_{b_R}^{\infty} \exp(-\mathrm{e}^{-(r-b_S)/a_S}) \frac{c_R}{a_R} \left(\frac{r - b_R}{a_R} \right)^{c_R - 1} \mathrm{e}^{-((r-b_R)/a_R)^{c_R}} \, \mathrm{d}r$$

This horrible integral cannot be computed analytically. There is in Matlab, however, a simple numerical integration routine, `trapz`, using the trapezium rule. To use it, one must define the integrand on a grid[4]:

```
>> aS=2.5; bS=55; aR=5; bR=70; cR=2;    % cR>=1
>> r=bR:0.01:150;
>> integrand=exp(-exp(-(r-bS)/aS)).*(cR/aR).*((r-bR)/aR).^(cR-1).*exp(-((r-bR)/aR).^cR);
```

Iit is a good rule to plot the integrand before performing a numerical integration.

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[3]The random variable $Z = h(Y_1; \ldots; Y_m; X_1; \ldots; X_n)$ is sometimes referred to as the *safety margin*.
[4]Here we have used that $c_R = 2 \ge 1$.

```
>> plot(r,integrand), grid on
```

From the plot it seems that 150 will do as an upper truncation limit; less would also have been sufficient. Now, let us calculate P_f by means of `trapz`:

```
>> Pf=1-trapz(r,integrand)
```

Question 3: Report the computed failure probability, P_f.

Simulation of P_f

Above you probably recognized the Gumbel distribution of S and the Weibull distribution of R. Since we know the parameters a_S, b_S, a_R, b_R, and c_R, we can simulate S and R (and hence $h(S;R)$) from these distributions. Simulating *many* times, $N = 200\,000$ say, we can estimate P_f frequentistically:

$$\widehat{P}_\mathrm{f} = \frac{\text{number of simulations (among the } N \text{ simulations) where } h(S;R) < 0}{N}$$

In Matlab[5]:

```
>> N=200000;
>> S=gumbrnd(aS,bS,1,N);
>> R=bR+wblrnd(aR,cR,1,N);
>> h=R-S;
>> Pfhat=sum(h<0)/N
```

Question 4: Do you understand what we did here? Compare with the value obtained from the numerical integration above! Repeat the simulation a couple of times to get a feeling for how stable the estimate is. Why is it not that clever to choose, for example, $N = 500$ (`N=500`)?

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4   Travel times

[6]A limousine operates from an airport, goes to town I, then to town II, and then returns to the airport as shown in Figure 1. Because of varied traffic conditions, the travel time required in each leg of the journey is random, with the following statistics:

Travel time	Mean	Standard deviation
T_1	40 min	10 min
T_2	15 min	5 min
T_3	50 min	10 min

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[5]Remember the routines `rand`, `randn`, `wblrnd`, `gumbrnd`, etc. from Computer Exercise 1 and 2. The function `gumbrnd` is in `labfiles` - it is our own function that is not available in the general Matlab package, see also comments in Computer Exercise 2.

[6]This example is taken from Ang, Alfredo H S – Tang, Wilson H: Probability Concepts in Engineering Planning and Design. – Volume II: Decision, Risk, and Reliability. — New York: John Wiley, 1984. — ISBN 0-471-03200-X.

We assume that the traffic conditions over the three legs of the journey can be considered *independent* (may this be realistic?). The scheduled time for each round trip around the loop is two hours (2 h = 120 min). Determine the probability that a round trip will not be completed on schedule.
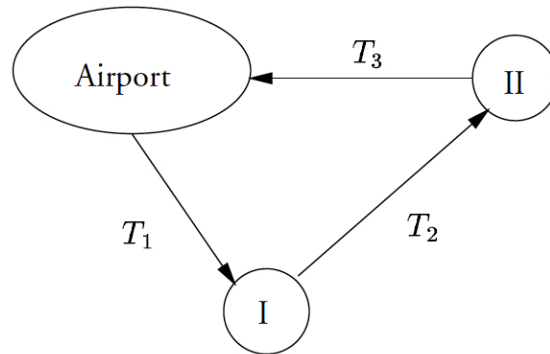

Figure 1: Travel-times problem

The failure function is given by

$$h(T_1, T_2, T_3) = 120 \text{ min} - (T_1 + T_2 + T_3).$$

Question 5: Assume that T_1, T_2 and T_3 are *normally distributed*. Hence, the probability of failure can be calculated explicitly. Write down the distribution of $T_1 + T_2 + T_3$.

From this it follows that

$$P_f = P(h(T_1, T_2, T_3) < 0) = P(T_1 + T_2 + T_3 > 120 \text{ min}) =$$

$$= 1 - P(T_1 + T_2 + T_3 \leq 120 \text{ min}) = 1 - \Phi\left(\frac{120 \text{ min} - 105 \text{ min}}{15 \text{ min}}\right) = 1 - \Phi(1)$$

This is calculated in Matlab by

```
>> Pf=1-normcdf(1)
```

Question 6: What is the probability of the failure?

Simulation of P_f

Similarly as in the previous example, we can estimate P_f by means of simulations.

```
>> N=10000;
>> T1=40+10*randn(1,N);
>> T2=15+5*randn(1,N);
>> T3=50+10*randn(1,N);
>> h=120-(T1+T2+T3);
>> Pfhat=sum(h<0)/N
```

Question 7: Repeat the simulations a couple of time and report the obtain values. Are the similar to the one obtained above by theoretical considerations.

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5 Spillway capacity

We will now study a civil-engineering problem: the capacity of spillway gates in dam constructions. This time we will rely on simulations only. We begin with an example: the Folsom Dam in California and what happened to it a couple of years ago. Thereafter we will present a model for the discharge capacity of a spillway of a certain kind, and a failure function will be presented. From assumed distributions we will then simulate in order to estimate the probability that the spillway capacity will be exceeded.

The Folsom Dam

The Folsom Dam in California is a concrete gravity structure on the American River. The dam has a structural height of 340 ft. The dam crest is at elevation 481 ft with length of 26 670 ft, and width of 36 ft. The maximum base width is 270 ft.

Normally, when a reservoir becomes too full – like after a heavy rainstorm – engineers open spillway gates, which allow the excess water to drain out of the reservoir at a controlled rate of speed. When these gates open suddenly and engineers lose the ability to control the flow, disaster can result.

On the 17th of July 1995, spillway gate No 3 of the Folsom Dam suffered a partial failure (Figure 2), which allowed an uncontrolled maximum flow of approximately 40 000 ft^3/s to pass the dam. Nearly 40 % of Folsom Lake drained out past the broken gate before it could be repaired. Fortunately, this release was well below the flow capacity of the river downstream from the dam, and there was no flooding outside the embankments[7].

A model, and simulating from it

[8]Insufficient spillway capacity to carry the inflow water during an extreme flood is obviously a major cause of dam failure. For a spillway of a certain kind, the *discharge* Q_c is given by

$$Q_c = KCLH^{3/2}$$

Here

$\begin{cases} C = \text{the discharge coefficient} \\ L = \text{effective length of spillway} \\ H = \text{head (maximum height) of water crest at the spillway (effects of velocity of approach are included)} \\ K = \text{correction for imperfections in the empirical formula} \end{cases}$

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[7]See http://www.pbs.org/wgbh/buildingbig/wonder/structure/folsom.html.

[8]This example is taken from Ang, Alfredo H S – Tang, Wilson H: Probability Concepts in Engineering Planning and Design. – Volume II: Decision, Risk, and Reliability. — New York: John Wiley, 1984. — ISBN 0-471-03200-X.

Figure 2: Failure of a spillway

The inflow rate at the spillway can be modelled by

$$Q_{\mathrm{L}} = RQ_{\mathrm{i}}$$

where Q_{i} is the *peak* flow upstream to the reservoir and R is the attenuation factor to account for the volume effect of the reservoir. Insufficient spillway capacity is per definition at hand when

$$Q_{\mathrm{c}} < Q_{\mathrm{L}}.$$

Assume that $K = 1,0$ is deterministic, and suppose that the remaining variables have the following statistics:

	Mean	Standard deviation	Distribution
C	3,85	0,27	Normal
L	93,4	5,6	Normal
H	12,0	0,72	Normal
R	0,7	0,1	Normal
Q_{i}	$9,1 \cdot 10^3$	$3,2 \cdot 10^3$	Gumbel

The random variables C, L, H, R, and Q_{i} are considered independent.

Question 8: Write down the failure function $h(C; L; H; R; Q_{\mathrm{i}})$ for the system.

As usual, let us estimate P_{f} from simulations in Matlab:

```
>> K=1.0;
>> EC=3.85; DC=0.27;
>> EL=93.4; DL=5.6;
>> EH=12.0; DH=0.72;
>> ER=0.7; DR=0.1;
>> EQi=9.1e3; DQi=3.2e3;
>> aQi=sqrt(6)*DQi/pi, bQi=EQi-0.57721566*aQi
```

> **Question 9:** Can you explain the last line of the above code?

```
>> N=2000;    % for example
>> C=EC+DC*randn(1,N);
>> L=EL+DL*randn(1,N);
>> H=EH+DH*randn(1,N);
>> R=ER+DR*randn(1,N);
>> Qi=wgumbrnd(aQi,bQi,[],1,N);
>> h=K*C.*L.*(H.^(3/2))-R.*Qi;
>> Pfhat=sum(h<0)/N
```

Repeat this a couple of times as usual. Plot the failure function `h` between the sets of simulations:

```
plot(1:N,h,'.'), grid on
```

> **Question 10:** Report simulated probabilities of failure. If the model represent a situation after each heavy rain, would you find the probability of failure acceptable?

6   MOSFET

A depletion-mode MOSFET (Metal-Oxide-Semiconductor Field-Effect Transistor) is a three-terminal electronic device. When an n-channel MOSFET is connected like this:
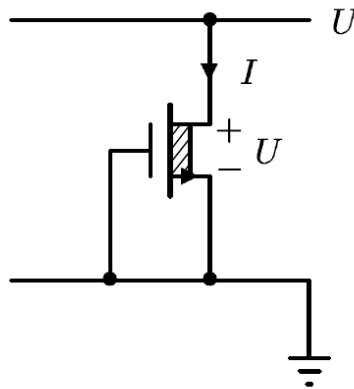
then it has the following voltage-current characteristic.

$$I = \begin{cases} A \cdot V_{\text{TR}}^2; & U > V_{\text{TR}} & \text{(Constant current region)} \\ = A \cdot (2V_{\text{TR}}U - U^2); & 0 < U < V_{\text{TR}} & \text{(Triode region)} \\ = \text{undefined?}; & U < 0 \end{cases}$$

Here U is the applied voltage[9], V_{TR} is a threshold voltage (always positive for n-channel MOSFETs), and A is the conductance parameter.

When current flows into the positive terminal of a passive device, electrical power is dissipated in the device as heat. This electrical power P is equal to the product of the port voltage and port current. For a multiport device, the total electrical power input is given by the sum of input power taken over all ports. The dissipated energy will increase the temperature of the device, which affects the properties of it. Every device has a maximum allowable operating temperature limit that must not be exceeded. In other words: there is a maximum electrical power limit P_{max}. In our case,

$$UI < P_{\mathrm{max}}$$

if the MOSFET is to work well. Assume that U, V_{TR}, and A are independent random variables:

U	Normal with mean 10 V and standard deviation 2 V
A	Log-normal with median 1 mA/V^2 and $\sigma = 0,2$
V_{TR}	Uniform between 3 V and 5 V

and that $P_{\mathrm{max}} = 300$ mW.

```
>> Pmax=300e-3;
>> N=20000;
>> EU=10; DU=2;
>> medianA=1e-3; sigma=0.2;
>> aVTR=3; bVTR=5;
>> U=EU+DU*randn(1,N);
>> A=medianA*exp(sigma*randn(1,N));
>> VTR=aVTR+(bVTR-aVTR)*rand(1,N);
>> I=zeros(1,N);
>> index1=find(U>=VTR);
>> index2=find(U<VTR);
>> I(index1)=A(index1).*VTR(index1).^2;                 % Constant current region
>> I(index2)=A(index2).*(2*VTR(index2).*U(index2)-U(index2).^2); % Triode region
>> h=Pmax-U.*I;
>> Pfhat=sum(h<0)/N
```

Question 11: Report simulated probabilities of failure (do few repetitions).

Question 12: Is $\mathsf{P}(U < 0)$ negligible? If $U < 0$ was not negligible, it is bad for the MOSFET, so let us consider this case to be a failure. Write down a failure function $h(P_{max}; U; A; V_{TR})$ with this extra condition.

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[9]i.e. the drain-source voltage