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Chalmers and GU

MVE550 Stochastic Processes and Bayesian Inference

Trial exam autumn 2018
Allowed aids: Chalmers-approved calculator.
Total number of points: 30. To pass, at least 12 points are needed

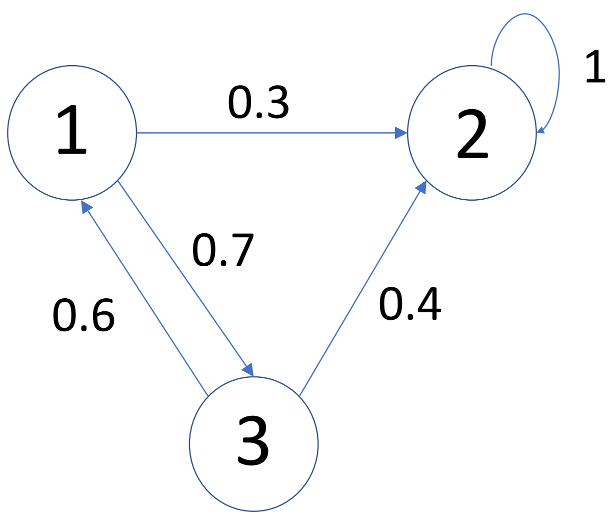

Figure 1: The chain for question 1.

1. (4 points) Consider the discrete-time Markov chain with state space $\{1, 2, 3\}$ and transition probabilities given in Figure 1.
   (a) What is the limiting distribution for this chain?
   (b) Write down its transition matrix, and compute its fundamental matrix F.
   (c) If a chain starts in state 1, what is the expected number of steps it will be in state 3?
2. (4 points) Consider an ergodic discrete-time Markov chain with a finite state space, transition matrix $P = [P_{ij}]$, and stationary distribution v.
   (a) If we simulate k independent Markov chains, with each chain having a different starting value at X_0, what is the probability that all chains are in the same state after n steps, as $n \to \infty$?

(b) Describe a way to set up the simulation so that each chain is still a realization from the Markov chain with the given transition matrix, and each chain starts with a different starting value, but the probability that all chains are in the same state after n steps approaches 1 as $n \to \infty$.

(c) Describe a type of sampling using the simulation method of (b), and explain why this is useful in this sampling type.

3. (3 points) Let X be a random variable with values in the set $\{0, 1, 2, \dots\}$ and let $G(s)$ be its probability generating function. The variance of X can be expressed in terms of derivatives of $G(s)$ evaluated at specific values for s. Derive the correct formula.

4. (4 points) Assume $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ. Assume each arrival is independently marked with M (with probability p) or with F (with probability $1 - p$). Let $N_t^{(M)}$ and $N_t^{(F)}$ be the number of events of type M or F, respectively, in the interval $[0, t]$. Prove that $N_t^{(M)}$ and $N_t^{(F)}$ are independent random variables and derive their distributions.

5. (5 points) Define a discrete-time continuous-valued Markov chain $X_0, X_1, X_2, \dots$, by defining $X_0 \sim \text{Normal}(0, 1)$ and for $k = 1, 2, \dots,$

$$X_k \sim \text{Normal}(aX_{k-1}, 1)$$

where a is a real parameter.

(a) Assume we use the prior $a \sim \text{Normal}(1, 1)$ for a. Find the posterior distribution for a given observations x_0 and x_1 of X_0 and X_1, respectively.

(b) Prove that the posterior for a given observations of $X_0, X_1, \dots, X_k$ is normal. You do not need to derive the parameters of the distribution.

6. (5 points) Consider a continuous-time Markov chain on the state space $\{1, 2, 3, 4\}$. The transition rates between the states are indicated in the transition rate graph in Figure 2.

(a) Is this chain ergodic? Why/why not?

(b) Write down the (infinitesimal) generator matrix Q for this chain.

(c) Let $v = (v_1, v_2, v_3, v_4)$ be the probability vector representing the staionary distribution for the continuous-time Markov chain. Write down four linear equations that the numbers v_1, v_2, v_3, v_4 need to satisfy, and which determine these numbers uniquely. (You do not need to solve the equations).

(d) Write down the transition matrix for the embedded Markov chain.

(e) Let $w = (w_1, w_2, w_3, w_4)$ be the probability vector representing the stationary distribution for the discrete embedded Markov chain. Write down four linear equations that the numbers w_1, w_2, w_3, w_4 need to satisfy, and which determine these numbers uniquely. (You do not need to solve the equations).

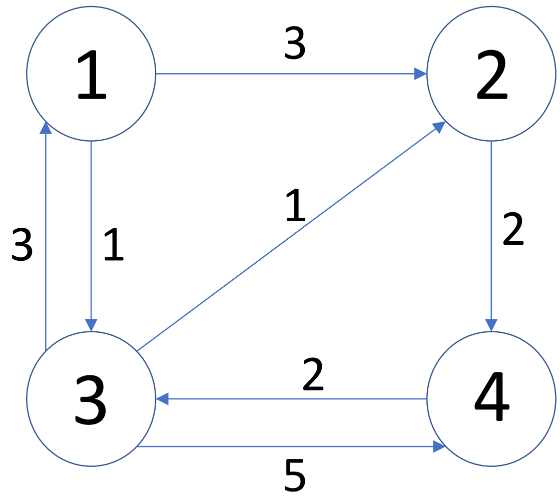

Figure 2: The chain for question 6.

7. (5 points) Assume $\{B_t\}_{t\geq 0}$ is the Brownian motion stochastic process.

   (a) Find the distribution of $B_1 + B_2 + B_3$.

   (b) Give the definition of a Gaussian process (as defined in Dobrow).

   (c) Let $a > 0$. For $n = 1, 2 \ldots,$, let M_n be the event consisting of those $t > 0$ such that there exists $t_1 < t_2 < \cdots < t_n < t$ with $B_{t_i} = a$ for $i = 1, \ldots, n$. Let T_n be the largest t smaller than or equal to all $t \in M_n$. Compute the distribution of $T_{100} - T_1$.

Appendix: Some probability distributions

The Beta distribution

If $x \geq 0$ has a Beta(α,β) distribution with $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

The Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Binomial(n, p) distribution, with n a positive integer and $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

The Exponential distribution

If $x \geq 0$ has an Exponential(λ) distribution with $\lambda > 0$ as parameter, then the density is

$$\pi(x \mid \lambda) = \lambda \exp(-\lambda x)$$

and the cumulative distribution function is

$$F(x) = 1 - \exp(-\lambda x).$$

The Gamma distribution

If $x > 0$ has a Gamma(α,β) distribution, with $\alpha > 0$ and $\beta > 0$, then the density is

$$\pi(x \mid \alpha\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

The Normal distribution

If the real x has a Normal distribution with parameters μ and σ^2, its density is given by

$$\pi(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

The Poisson distribution

If $x \in \{0, 1, 2, \ldots\}$ has a Poisson(λ) distribution, with $\lambda > 0$, then the probability mass function is

$$e^{-\lambda}\frac{\lambda^x}{x!}.$$