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Suggested solutions for
MVE550 Stochastic Processes and Bayesian Inference
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1. (a) We get

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

(b) For any random walk on a weighted undirected graph, the stationary distribution can be read off the graph. Note that the sum of all weights is 10. Thus

$$\begin{aligned} v &= \left(\frac{1+2}{2\cdot 10}, \frac{2+1+1+2}{2\cdot 10}, \frac{1+1+2}{2\cdot 10}, \frac{1+1+2}{2\cdot 10}, \frac{1+2}{2\cdot 10}\right) \\ &= (0.15, 0.3, 0.2, 0.2, 0.15) \end{aligned}$$

(c) Let v be the probability vector representing the stationary distribution. Then the chain is time-reversible if and only if, for all i and j,

$$v_i P_{ij} = v_j P_{ji}.$$

Let w_{ij} denote the weight on the line connecting state i and state j. Then, for the random walk, we have

$$P_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$$

and

$$v_i = \frac{\sum_k w_{ik}}{\sum_s \sum_k w_{sk}}.$$

Thus

$$\begin{aligned} v_i P_{ij} &= \frac{\sum_k w_{ik}}{\sum_s \sum_k w_{sk}} \cdot \frac{w_{ij}}{\sum_k w_{ik}} = \frac{w_{ij}}{\sum_s \sum_k w_{sk}} \\ v_j P_{ji} &= \frac{\sum_k w_{jk}}{\sum_s \sum_k w_{sk}} \cdot \frac{w_{ji}}{\sum_k w_{jk}} = \frac{w_{ji}}{\sum_s \sum_k w_{sk}}. \end{aligned}$$

As $w_{ij} = w_{ji}$ for all i and j, we have time-reversibility.

2. (a) We get

$$\begin{aligned}
\pi(p \mid x) &\propto_p \pi(x \mid p)\pi(p) \\
&\propto_p \text{Geometric}(x; p) \cdot \text{Beta}(p; \alpha, \beta) \\
&\propto_p p(1-p)^{x-1} p^{\alpha-1}(1-p)^{\beta-1} \\
&\propto_p p^{\alpha+1-1}(1-p)^{\beta+x-1-1}
\end{aligned}$$

Thus $p \mid x \sim \text{Beta}(\alpha+1, \beta+x-1)$.

(b) We get

$$\begin{aligned}
\pi(x) &= \frac{\pi(x \mid p)\pi(p)}{\pi(p \mid x)} \\
&\propto_x \frac{\text{Geometric}(x; p)}{\text{Beta}(p; \alpha+1, \beta+x-1)} \\
&\propto_x \frac{p(1-p)^{x-1}}{\frac{\Gamma(\alpha+1+\beta+x-1)}{\Gamma(\alpha+1)\Gamma(\beta+x-1)} p^{\alpha+1-1}(1-p)^{\beta+x-1-1}} \\
&\propto_x \frac{\Gamma(\beta+x-1)}{\Gamma(\beta+x+\alpha)}
\end{aligned}$$

Thus $f(x) = \frac{\Gamma(\beta+x-1)}{\Gamma(\beta+x+\alpha)}$. When α is an integer, this corresponds to $f(x) = \frac{1}{(\beta+x-1)(\beta+x)\cdots(\beta+x+\alpha-1)}$.

3. (a) The fundamental matrix is

$$F = (I - Q)^{-1} = (1 - (1-p))^{-1} = p^{-1} = 1/p$$

i.e., the 1×1 matrix with the single element $1/p$.

(b) The expected number of steps until absorbtion can be found from the fundamental matrix, i.e., it is $1/p$.

(c) Let X denote the number of steps until absorbtion. Then we can read from the definition of P that, for $k = 1, 2, \ldots,$

$$\mathrm{P}(X = k) = p(1-p)^{k-1}.$$

This means that $X \sim \text{Geometric}(p)$. From the appendix we have that $\text{Var}[X] = (1-p)/p^2$, so this is the answer.

4. (a) The offspring distribution has expectation

$$0 \cdot \frac{1}{8} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{3}{8} = \frac{5}{4}$$

Thus

$$\mathrm{E}(Z_5) = \left(\frac{5}{4}\right)^5 = \frac{3125}{1024} = 3.051758$$

(b) We get

$$G(s) = \mathrm{E}(s^X) = \frac{1}{8} + \frac{1}{2}s + \frac{3}{8}s^2$$

(c) We get

$$\begin{aligned}
G(s) &= s \\
\frac{1}{8} + \frac{1}{2}s + \frac{3}{8}s^2 &= s \\
3s^2 - 4s + 1 &= 0 \\
(s-1)(3s-1) &= 0
\end{aligned}$$

Thus the smallest positive root of $G(s) = s$ is 1/3, which is the probability of extinction.

5. The algorithm starts with selecting an initial real value $x^{(0)}$. Then, for $i = 1, 2, \ldots,$, the algorithm generates $x^{(i)}$ as follows:

- Simulate a proposed value $y \sim \text{Normal}(x^{(i-1)}, \sigma_a^2)$.
- Compute the acceptance probability:

$$\begin{aligned}
p &= \min\left(1, \frac{\pi(y)q(x^{(i-1)} \mid y)}{\pi(x^{(i-1)})q(y \mid x^{(i-1)})}\right) \\
&= \min\left(1, \exp\left(-(y + \sin y)^2 + (x^{(i-1)} + \sin x^{(i-1)})^2\right)\right).
\end{aligned}$$

NOTE: The quotient above is *not* $\frac{\pi(y)q(y|x^{(i-1)})}{\pi(x^{(i-1)})q(x^{(i-1)}|y)}$.

- With probability p, set $x^{(i)} = y$, otherwise, set $x^{(i)} = x^{(i-1)}$.

The distribution of the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ will now converge to a distribution with density $\pi(x)$.

6. For $i = 1, 2, \ldots$, let X_i be the holding time between arrival $i-1$ and arrival i. Then all the X_i are independent and $X_i \sim \text{Exponential}(\lambda)$. Also $S_n = \sum_{i=1}^n X_i$ and $S_m - S_n = \sum_{i=n+1}^m X_i$.

(a) We get

$$\begin{aligned}
\mathrm{E}(S_m - S_n) &= \mathrm{E}\left(\sum_{i=n+1}^m X_i\right) = \sum_{i=n+1}^m \mathrm{E}(X_i) = \frac{m-n}{\lambda} \\
\mathrm{var}(S_m - S_n) &= \mathrm{var}\left(\sum_{i=n+1}^m X_i\right) = \sum_{i=n+1}^m \mathrm{var}(X_i) = \frac{m-n}{\lambda^2}.
\end{aligned}$$

(b) We get

$$
\begin{aligned}
\operatorname{corr}(S_m, S_n) &= \frac{\operatorname{cov}(S_m, S_n)}{\sqrt{\operatorname{var}(S_m) \cdot \operatorname{var}(S_n)}} \\
&= \frac{\operatorname{cov}\left(S_n + \sum_{i=n+1}^{m} X_i, S_n\right)}{\sqrt{\operatorname{var}(S_m) \cdot \operatorname{var}(S_n)}} \\
&= \frac{\operatorname{cov}\left(S_n, S_n\right) + \operatorname{cov}\left(\sum_{i=n+1}^{m} X_i, S_n\right)}{\sqrt{\operatorname{var}(S_m) \cdot \operatorname{var}(S_n)}} \\
&= \frac{\operatorname{cov}(S_n, S_n)}{\sqrt{\operatorname{var}(S_m) \cdot \operatorname{var}(S_n)}} \\
&= \frac{\sqrt{\operatorname{var}(S_n)}}{\sqrt{\operatorname{var}(S_m)}} \\
&= \sqrt{\frac{n/\lambda^2}{m/\lambda^2}} = \sqrt{\frac{n}{m}}.
\end{aligned}
$$

7. (a) Ordering the states of the hair salon as

   i. No customers.
   ii. Only A working.
   iii. Only B working.
   iv. Both A and B working but no-one waiting.
   v. Both A and B working and one person waiting.

   we get (using hours as the unit of time)

$$
Q = \begin{bmatrix} -3 & 0 & 3 & 0 & 0 \\ 3 & -6 & 0 & 3 & 0 \\ 2 & 0 & -5 & 3 & 0 \\ 0 & 2 & 3 & -8 & 3 \\ 0 & 0 & 0 & 5 & -5 \end{bmatrix}.
$$

   (b) Let $v = (v_1, v_2, \ldots, v_5)$ be the stationary distribution for the process. Then the answer to the question is given by v_2. We know that $vQ = 0$ and that $\sum_{i=1}^{5} v_i = 1$. These equations represent 6 equations for the 5 unknown components of v. The equations are

$$
\begin{aligned}
-3v_1 + 3v_2 + 2v_3 &= 0 \qquad (1) \\
-6v_2 + 2v_4 &= 0 \qquad (2) \\
3v_1 - 5v_3 + 3v_4 &= 0 \qquad (3) \\
3v_2 + 3v_3 - 8v_4 + 5v_5 &= 0 \qquad (4) \\
3v_4 - 5v_5 &= 0 \qquad (5) \\
v_1 + v_2 + v_3 + v_4 + v_5 &= 1 \qquad (6)
\end{aligned}
$$

To find v_2 we need to solve this system. We may in fact remove any of the equations (1) through (5).

8. From the definition of the exponential matrix and using $A = SDS^{-1}$, we get

$$e^A = S e^D S^{-1}$$

Thus

$$\begin{aligned}
\det(e^A) &= \det(S e^D S^{-1}) = \det(S)\det(e^D)\det(S^{-1}) \\
&= \det(S)\det\begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^{1/2} & 0 \\ 0 & 0 & e^{1/3} \end{bmatrix}\det(S)^{-1} \\
&= e^1 e^{1/2} e^{1/3} = e^{11/6} = 6.254701.
\end{aligned}$$