Lecture 2: Random fields

Statistical Image Analysis

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March 21, 2018

Finite dimensional distributions

- Let $D \subseteq \mathbb{R}^d$ be a spatial domain of interest.
- $X(\mathbf{s})$, $\mathbf{s} \in D$, can be thought of as a function-valued random variable, with realisations $X(\mathbf{s}, \omega)$ where $\omega \in \Omega$, and Ω is some abstract sample space.
- Fixing a set of locations $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$,

$$\mathbf{X} = (X(\mathbf{s}_1), \ldots, X(\mathbf{s}_n))^T$$

  is a multivariate random variable.
- The distribution of the process is given by the collection of the finite dimensional distributions

$$F(x_1, \ldots, x_n; \mathbf{s}_1, \ldots, \mathbf{s}_n) = \mathsf{P}(X(\mathbf{s}_1) \leq x_1, \ldots, X(\mathbf{s}_n) \leq x_n)$$

  for all $n < \infty$ and every set of locations $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$.
- Kolmogorov existence theorem: The model is valid if the family of finite-dimensional distributions is consistent under reorderings and marginalizations (see Billingsley 1986).

Random fields

- We have measurements $y_i, \ldots, y_n$ taken at some spatial locations $s_1, \ldots, s_n$.
- Given that we also have some explanatory variables $B_1, \ldots, B_K$, we could use a regression model

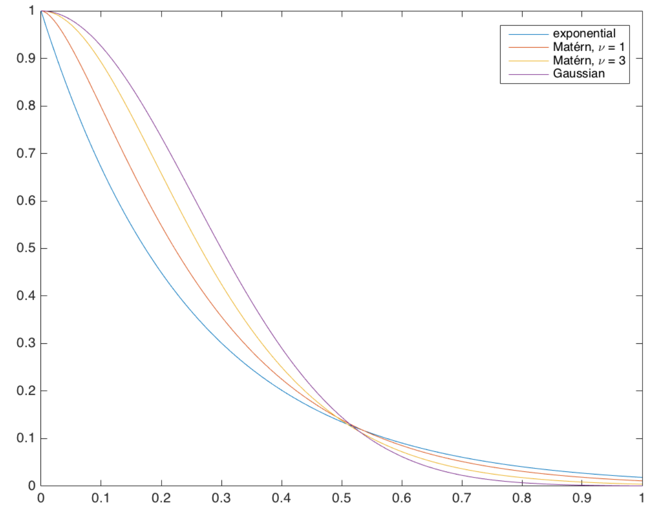
$$Y_i = \sum_{k=1}^{K} B_k(s_i)\beta_k + \varepsilon_i, \quad \varepsilon_i \sim \mathsf{N}(0, \sigma^2)$$

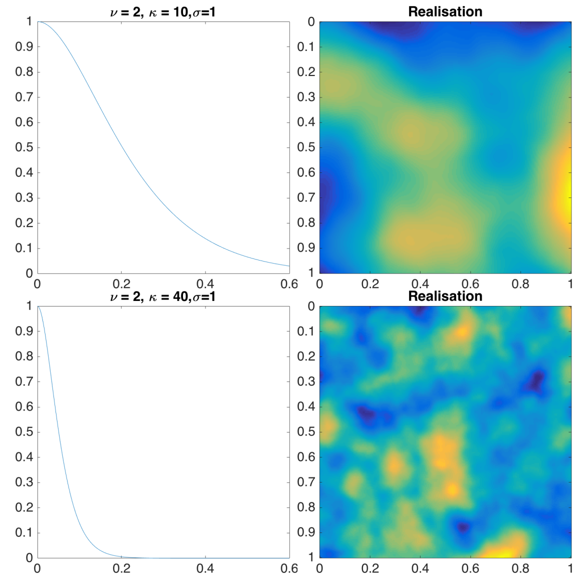
- The explanatory variables can often not capture all dependence for spatial data.
- Therefore, we would like to capture this additional dependence through a random field $X(s)$ in the model,

$$Y_i = \sum_{k=1}^{K} B_k(s_i)\beta_k + X(s_i) + \varepsilon_i \,.$$

- Today we will see how we can define this quantity.

Matérn covariances





Compactly supported covariance functions

- Euclid's hat covariance function:

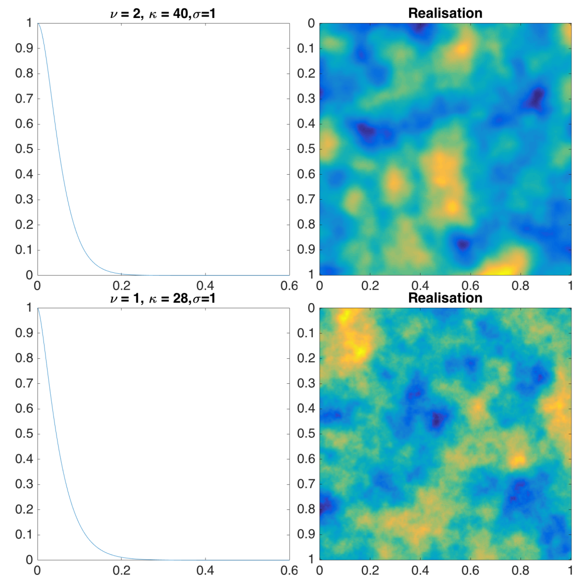
$$r_0(h) = \begin{cases} \sigma^2 I_{\frac{n+1}{2},\frac{1}{2}}(1 - h^2/\theta^2) & h \leq \theta \\ 0 & h > \theta \end{cases}$$

  where

$$I_{\frac{n+1}{2},\frac{1}{2}}(x) = \frac{\int_0^x \sqrt{t^{n-1}(1-t)^{-1}}dt}{\int_0^1 \sqrt{t^{n-1}(1-t)^{-1}}dt}$$

  is the regularized incomplete beta function.
- It is a valid covariance for $\mathbb{R}^d$ for $n \geq d$.
- $n = 3$ gives us the popular spherical covariance function:

$$r_0(h) = \begin{cases} \sigma^2(1 - \frac{3}{2}\frac{h}{\theta} + \frac{1}{2}\frac{h^3}{\theta^3}), & h \leq \theta \\ 0 & h > \theta \end{cases}$$



Euclid's hat with $\theta = 1$