Statistical Image Analysis

Lecture 5: Gaussian Markov random fields

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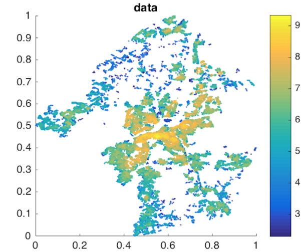
Example project 2



Data from the Spatial Morphology Group at Chalmers

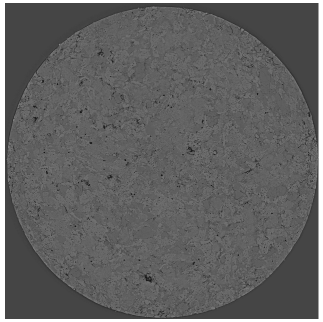
- For city planning it is important to know how the structure of the city affects things such as population density, and housing prices.
- In this project, the aim is to estimate how spatial models using various network measures can predict population denstiy or housing prices.

Example project 1

Data from Agroväst Livsmedel AB

- To increase the quality of animal fodder, clover is commonly grown alongside foraging grasses. A healthy balance is around $20-30\%$ clover.
- It is important that that farmers can get reliable estimates of the proportion, which currently is done manually.
- The aim of this project is to be able to estimate the clover proportion directly from images of the foraging grasses.

Example project 3



Data from AstraZeneca

- For the production of medical tablets, it is important to know how the manufacturing process affects the composition.
- To do this, one first needs to be able to identify the different components in the tablet based on micro-CT images.
- The goal of this project is to design a method for image segmentation of such images.

Example project 4

Data from Department of Historical Studies at GU

- The Swedish rock art archive contains several thousand images of rock art.
- To simplify analysis of such images, it is of interest to design algorithms for automatic segmentation and classification.
- In this project, you could either focus on image segmentation or classification.

Implementation aspects

Consider the problem of sampling. Two important aspects are

1. The RAM memory required, which is dominated by the memory required to store $\boldsymbol{\Sigma}$, which has $\mathcal{O}(N^2)$ unique elements.
2. The computation time for performing the necessary steps: Compute $\boldsymbol{\Sigma}$, compute the Cholesky factorization $\boldsymbol{\Sigma} = \mathbf{R}^T\mathbf{R}$, solve $\mathbf{x} = \mathbf{R}^T\mathbf{e}$ with $\mathbf{e} \sim \mathsf{N}(\mathbf{0}, \mathbf{I})$. This requires $\mathcal{O}(N^3)$ flops.

Assume that $\mathbf{x}$ is an image of size $N = n \times n$. The following table gives some results for the sampling on a standard laptop.

	time (s)	Memory (MB)
n = 50	1.1	47.7
n = 100	23.4	762.9
n = 150	272.5	3862.4

An image of size 150×150 is not a very large image!

Gaussian random fields

So far, we have looked at models

$$Y_i = \mathbf{B}(\mathbf{s}_i)\boldsymbol{\beta} + X(\mathbf{s}_i) + \varepsilon_i, \quad i = 1, \ldots, N$$

where $\varepsilon_i \sim \mathsf{N}(0, \sigma_e^2)$ and $X(\mathbf{s})$ is a Gaussian random field.

- The data vector $\mathbf{Y} = (Y_1, \ldots, Y_N)^T$ has distribution $\mathsf{N}(\mathbf{B}\boldsymbol{\beta}, \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_X + \sigma_e^2\mathbf{I}$.
- log-likelihood: $\ell(\mathbf{Y}; \boldsymbol{\beta}, \boldsymbol{\theta}) = \frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}(\mathbf{Y} - \mathbf{B}\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{B}\boldsymbol{\beta})$.
- Kriging: $\mathsf{E}(\mathbf{Y}_0|\mathbf{Y}, \boldsymbol{\beta}, \boldsymbol{\theta}) = \mathbf{B}(\mathbf{s}_0)\boldsymbol{\beta} + \mathbf{r}\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{B}\boldsymbol{\beta})$, where $\mathbf{r}_i = \mathsf{C}(Y_0, Y_i)$.
- Sampling: $\mathbf{Y}_s = \mathbf{B}\boldsymbol{\beta} + \mathbf{R}^T\mathbf{e}$, where $\mathbf{e} \sim \mathsf{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{R}^T\mathbf{R} = \boldsymbol{\Sigma}$ is the Cholesky factorization.

Gaussian Markov random fields

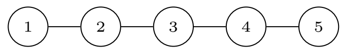
- A Multivariate Gaussian random variable is said to be a GMRF with respect to the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if

$$\pi(x_i|x_{-i}) = \pi(x_i|x_{N_i})$$

  where $N_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$.
- Example: The AR(1) process defined by

$$x_0 \sim \mathsf{N}(0, (1+\alpha^2)^{-1})$$
$$x_i = \alpha x_{i-1} + \varepsilon_i, \quad \varepsilon_i \sim \mathsf{N}(0,1)$$

  for $\alpha \in (-1, 1)$ is a GMRF with respect to the graph



Computation times for a GMRF

Assume that $\mathbf{x}$ is an image of size $N = n \times n$, chosen as a GMRF specified using the stencil

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

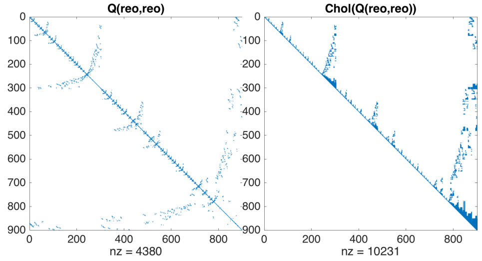
Let us now sample $\mathbf{x}$ and measure

1. The RAM memory required.
2. The computation time for performing the necessary steps.

The following table gives some results for the sampling on a standard laptop.

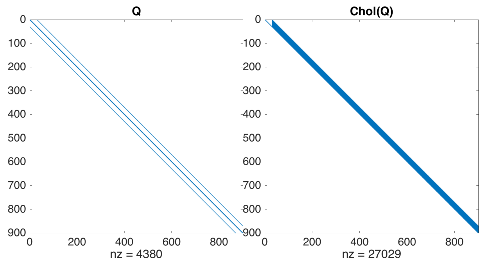
	time (s)	Memory (MB)
n = 50	0.012	0.21
n = 100	0.054	0.83
n = 150	0.177	1.88

Sparsity using reorderings



- Finding the optimal reordering is an NP-hard problem, but there are many fast methods for finding good reorderings.
- The approximate minimum degree (AMD) reordering is generally a good option.
- The images above are obtained with `reo = amd(Q)` in Matlab.
- If you use reorderings, remember to also reorder the observations, covariates, etc. using the same reordering.

Sparsity of $\mathbf{Q}$ and $\mathbf{R}$



- The crucial aspect of computations with GMRFs is that the Cholesky factor $\mathbf{R}$ is sparse.
- However, it is often less sparse than the precision matrix $\mathbf{Q}$. The additional non-zero nodes is usually called fill-in.
- We can reduce the fill-in by reordering the nodes.