Statistical Image Analysis

Lecture 9: Image segmentation using MRFs

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Example of discrete MRFs

A general MRF for lattices with first-order neighborhoods:

$$p(\mathbf{z}) = \frac{1}{Z}\exp\left(\sum_i \alpha_{z_i} + \frac{1}{2}\sum_i \sum_{j\in N_i} \beta_{z_i,z_j}\right).$$

Here $\{\alpha_1,\ldots,\alpha_K\}$ determines the prior probabilities for the K classes and $\beta_{k,l}$ determines the interaction between class k and class l.

Common simplifications include assuming that

$$\beta_{k,l} = \begin{cases}\beta_k & \text{if } l = k\\ 0 & \text{otherwise}\end{cases} \quad \text{or} \quad \beta_{k,l} = \begin{cases}\beta & \text{if } l = k\\ 0 & \text{otherwise}\end{cases}$$

Markov random field mixture models

- Hierarchical model for pixel values given classes:

$$\pi(\mathbf{Y}_i|z_i = k) \sim \mathsf{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

$$\pi(z_i) = \begin{cases}\pi_1 & \text{if } z_i = 1\\ \pi_2 & \text{if } z_i = 2\\ \vdots & \\ \pi_K & \text{if } z_i = K\end{cases}$$

- Assuming independence between the pixels is not realistic!
- In a Markov random field mixture model, we use the model

$$\pi(\mathbf{Y}_i|z_i = k) \sim \mathsf{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$
$$\mathbf{z} \sim \pi(\mathbf{z})$$

  here $\mathbf{z} = (z_1,\ldots,z_n)$ is a random field that takes values in $\{1,\ldots,K\}$, with density $\pi(\mathbf{z})$.
- Spatial dependencies modeled through $\pi(\mathbf{z})$.

Conditional distributions

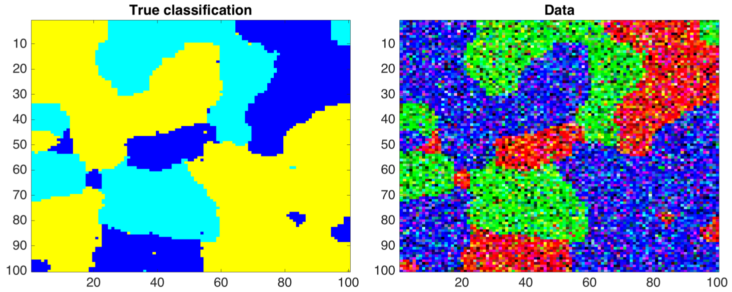
- The normalizing constant Z is intractable. However, the conditional distributions are simple:

$$p(z_i|\mathbf{z}_{-i}) = \frac{\exp(\alpha_{z_i} + \beta\sum_{j\in N_i} 1(z_j = z_j)}{\sum_k \exp(\alpha_k + \beta\sum_{j\in N_i} 1(z_j = k)}.$$

- Since we have simple conditional distributions, we can sample the field using Gibbs sampling.
- Since we cannot compute Z, likelihood-based inference is difficult.
- A solution to this problem is to instead work with the pseudo-likelihood

$$\pi_p(\mathbf{z}) = \prod_i \pi(z_i|\mathbf{z}_{-i})$$

Example data



Parameters:

$$\alpha_k = \frac{1}{3} \qquad \beta = 2 \qquad \mathbf{\Sigma}_k = 0.2\mathbf{I}$$

$$\boldsymbol{\mu}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \boldsymbol{\mu}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \boldsymbol{\mu}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$